Topology PS1

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Question 1

Let τ_l be the lower limit Topology on $\mathbb{R}$. Notice that $[0, 1)$ is in the lower limit topology (take $a = 0$ and $b = 1$ then), Thus evidently the interval is open in τ_l.

Now taking the complement of the interval, in $\mathbb{R}$, we get the set

$$(-\infty, 0) \cup [1, \infty)$$

Thus, since both $(-\infty, 0)$ and $[1, \infty)$ are in τ_l then it must be that there union is in the topology as well. Therefore, $[0, 1)$ is in closed in the lower limit topology.

Note: One may be confused why it is that the interval $(-\infty, 0)$ and $[1, \infty)$ are in the lower limit topolgy. This is sustained on the properties since the intervals could be composed from a union of infinite intervals from τ_l and so are also in the topology. More rigorously:

$$(-\infty, 0) = \bigcup_{i=1}^{\infty} [-i, 0) \subset \tau_l$$

$$[1, \infty) = \bigcup_{i=2}^{\infty} [1, i) \subset \tau_l$$

Question 2

a)

$$A(n : a) = \{a + nq | q \in \mathbb{Z}\}$$
$$\beta = \{A(n : a) | a, n \in \mathbb{Z} - \{0\}\}$$

Proving that β generates $\mathbb{Z}$:

- Given $t \in \mathbb{Z}$, then pick $B_t \in \beta$ s.t.

$$B_t = A(1 : t - 1)$$

Then $t \in B_t$

- Given $t \in \mathbb{Z}$ st $t \in A(n' : a') \cap A(n : a) \Rightarrow \exists q, q' st$:

$$\begin{aligned} a' + n'q' &= a + nq \\ \text{then let } \gamma &= n'n \\ \text{and let } B' &= A(\gamma, t) \\ &= \{t + \gamma q | q \in \{Z\}\} = \{t + n'nq | q \in \{Z\}\} \\ &= \{t + n'nq | q \in \{Z\} - 0\} \cup \{t\} \end{aligned}$$

Therefore we have that $t \in B' \subset A(n' : a') \cap A(n : a)$

b)

Proving that $A(n : a)$ is both open and closed in the topology generate by β on $\mathbb{Z}$, represented by τ_β:

- Trivially, since the τ_β is generated by the set of all sets $A(n : a)$, by construction the set $A(n : a)$ is open in τ_β
- Since τ_β is topology generated by the sets $A(n : a)$, its evident that $A(n : a)$ is closed in τ_β. However, formally one can check this:

$$\mathbb{Z} - A(n : a) = \bigcup_{j=1}^{a-1} A(j : a)$$

And since this is a union of open sets in τ_β, then we know that the $A(n : a)^c$ is closed in this topological space.

c)

Given that there would be finitely many primes, the the compliments of the union of these sets is equal to intersection of their compliments:

$$\left(\bigcup_p A(p : o) \right)^c = \bigcap_p (A(p : o))^c$$

And since each complement is open in τ_β then we have that their finite intersection is also open. Therefore, the union of the sets is closed.

d)

By looking at the compliment of the union of all primes we get that:

$$\left(\bigcup_p A(p:o)\right)^c = \mathbb{Z} - \left(\bigcup_p A(p:o)\right) = \{-1, 1\}$$

This is a finite set in a topology generated by infinite sets. Therefore we know that in fact the compliment of this set is not closed, which implies that the assumption that there are finite primes is false.

Question 3

Let τ_d be the topology on $\mathbb{Q}$ generated by the metric $d(x,y) = |x-y|$.

Pick the ball $P \subset \mathbb{Q}$ s.t. $P = B(0,\pi) = \{x \in \mathbb{Q} | d(x,0) \leq \pi\}$.

Proving that P is open and closed, respectively :

1. Notice that P is an open ball in $\mathbb{Q}$ and so it must be in τ_d. And so P is open in this topological space by construction.
2. Notice that:

$$\begin{aligned} \mathbb{Q} - P &= \mathbb{Q} - B(0,\pi) \\ &= \{x \in \mathbb{Q} | d(x,o) > \pi\} \\ &\subset \bigcup_{d(x,0)>\pi} B(x, \frac{d(x,\pi)}{2}) \\ &\subset \tau_d \end{aligned}$$

   This is a union of open balls of balls in $\mathbb{Q}$ where each ball is in τ_d and so the union of these sets is in τ_d. Therefore P is closed in τ_d

Question 4

a)

An example topological space in which not every one point set is closed:

$$X = \{a, b, c\}$$
$$\tau_X = \{\o, X, \{a\}, \{a, b\}\}$$
$$\Rightarrow \{a\} \text{ not closed in } \tau_X$$

b)

Let τ_H be the Hausdorff topological space of the set X. Assume $x \in X$ is not closed in τ_H

$$\Rightarrow X - x \text{ not open}$$
$$\Rightarrow \nexists U \in \tau_H s.t. X - x = U$$
$$\Rightarrow \forall U \in \tau_H, x \in U$$

This is a contradiction, since by construction we know that $\forall x, y \in X, \exists U, V \in \tau_H$ s.t. $x \in U, y \in V$ and $U \cap V = \o$ Thus, we have that:

$$\Rightarrow \forall y \in X - x$$
$$\Rightarrow \exists V_y \in \tau_H \text{ s.t. } y \in V_y \text{ and } x \notin V_y$$
$$\Rightarrow X - x \subset \bigcup_{y \in X, y \neq x} V_y \subset \tau_H$$

c)

Given a finite set X and the Hausdorff Topology on X, represented by τ_{HX} , we get that :

$\forall x_i \in X = \{x_1, x_2, \dots, x_n\}$ then by construction and properties of finite intersections in a topology $\exists U_i \in \tau_{HX}$ s.t. $U_i = \bigcap_{i \neq j} U_{i_j}$ where $x_i \in U_{i_j} \in \tau_{HX}, x_j \in U_j \in \tau_{HX}$ and $U_{i_j} \cap U_j = \o$ and $(i \neq j)$

Then we have that:

$$U = \{U_1, U_2, \dots, U_n\} s.t. U_i \cap U_j = \o \text{ when } i \neq j \text{ and } \forall x_p \in X, x_p \in U_p \subset U$$
$$\Rightarrow U \text{ generates } \tau_{discrete}$$
$$\Rightarrow \tau_{discrete} \subseteq \tau_{HX}$$

However $\tau_{discrete}$ is the most fine topology on X, therefore $\tau_{discrete} = \tau_{HX}$

This results suggest that for each x_i in X there is a set $\{U_i\}$ in τ_{HX} that only contains the single element, and so the set $U = \{U_1, U_2, \dots, U_n\}$ contains each element in X and are each disjoint with all other sets in U. And so, due to this property, it is evident that $U \in \tau_{HX}$ and is also the basis for the Discrete topology on X and so it must be that the Hausdorff topology is finer or as fine as the discrete topology. However, since, by definition, the discrete topology on a set is the most fine topology on the given set, it must be that the Discrete topology and the Hausdorff topology on a finite set are equivalent.