Homework 1 STA457

Shawn Unger - 1001261350
University of Toronto

Winter 2017

Contents

1 Question 1

a) **Plot the correlogram and periodogram of the orginal data (i.e. dollar).**
See Figure 1 in Appendix

b) **Plot the correlogram and periodogram of the first differences.**
See Figure 1 in Appendix

c) **Comment on the results obtained in parts (a) and (b). In particular, how are the correlograms and periodograms different? A simple model for the logarithm of exchange rates is a random walk – are the correlograms and periodograms in (a) and (b) consistent with this model?**
Comments on results for Q(1a):

(a) Periodogram
- We observe a decreasing trend in the spectrum of the periodogram
- There seems to be random movement along the center trend
- Max near 0

(b) Correlogram
- Slowly decaying trend
- Greater then zero for all lags

Comments on results for Q(1b)

(a) Periodogram
- stays around the mean for all frequencies
- random behaviour
- No obvious peaks

(b) Correlogram
- Decays very quickly to 0

Difference between Q1a and Q1b

	$Question(1a)$	$Question(1b)$
Periodogram	- Decreasing trend - Max at 0 frequency	- Constant trend - No obvious peaks
Correlogram	-Slowly montonically decaying and constantly over 0	-Decays very quickly to 0

These characteristics and differences between the model before and after we took the first difference. Evidently, after we took the first difference, the model start looking a lot like white noise. Moreover, since a random work is built on the usage of the value of the event prior plus white noise, taking the first difference would lead to such an effect. Moreover, the fact that the periodogram has a max at frequency 0 and has a decreasing trend with time corresponds to the features of white noise, since the model would have a high correlation with smaller frequencies, which would diminish as the size of the frequency increases due to the effect of the white noise.

d) **Now look at the correlogram and periodogram of the absolute values of the first differences (i.e. abs(returns)). Comment on the differences between the results for returns and abs(returns), in particular, with respect to the applicability of the random walk model.**

See Figure 2 in Appendix for plots.
Commentary:

- There exists a max point in the Periodogram of the absolute data at 0, while there is no evident peaks in the original return plot
  - This is an argument against our Random walk hypothesis, since if the returns were pure white noise, then it should not have any peaks in its spectrum (since the behavior is random and thus should be independent from time and prior values)
- Both Periodogram plots still resemble white noise
  - This is a parallel to our argument that the model resembles a random walk, since after taking the first difference, the model should resemble White Noise, which has random behaviour.
- The Correlogram of the absolute plot decreases to around 0.1 very quickly, and stay s around there with a few peeks around a lag of 14 and 20, while the original return plot decreases to 0 very quickly with little changes from 0.
  - this goes against the argument that suggest a resemblance to random walk since, if this was white noise, it should have a similar Correlogram to the original returns plot. i.e. quickly going to 0 ACF and staying there.

2 Question 2

Average monthly concentrations of carbon dioxide (CO2) from March 1958 to December 2016 at Mauna Loa volcano in Hawaii are given in the file CO2.txt.

a) **Plot the periodogram of the time series. At what frequencies are there peaks? To which features of the time series do these peaks correspond?**

See Figure 3 in Appendix for plots.
There are peaks at the frequencies 0, 0.0833,0.1667,0.2500, and 0.3444, which corresponds with every $12n$ months (where $n \in 0, 1, 2, 3, 4$, respectively)

b) **An estimate of the trend is given in the file CO2-trend.txt. Subtract the trend from the original data and look at the periodogram of the detrended data. Comment on the differences between the periodograms in parts (a) and (b). (It is useful here to overlay the two periodograms on the same plot.) In particular, how effective is the detrending in emphasizing the seasonality in the data?**

See Figure 3 in Appendix for plots.
Observational Commentary

- The peaks found in the Periodogram are quite similar, except the detrended Periodogram does not have a peak at frequency 0.
  - this is happens because evidently the trend relied on the directly prior events, and so removing it allows us to focus more on the seasonality.
- There is an evident oscillation in both plots, with a decrease in spectrum for the majority of $12n^\star$ period, and then a hike up (peak) when frequency gets closer to frequency of $12n^\star$ (when $n^\star = n\backslash\{0\}$). However, the trend decays slower in the original plot versed the differenced plot which seems to have an only slightly declining mean around which the oscillating behaviors occurs.
  - This shows evidence of seasonal effect in both plots, however, the detrended plot emphasises greater the seasonality effect, and still shows a lower spectrum over time, which makes sense given the effect of white noise effect over time. However, the detrending does allow us to see a slower decay in the trend of the data, while at the same time increasing the visibility of cyclicality in the data (increasing the prominence of the peaks and troughs), as well as eliminates the peak at frequency 0, and thus makes the seasonality easier to observe.

3 Question 3

a)

$$
\begin{aligned}
X(\omega)Y(\omega) &= \sum_{t=-\infty}^{\infty} x_t e^{i2\pi wt} \sum_{s=-\infty}^{\infty} y_s e^{i2\pi ws} \\
&= \sum_{t=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} x_t y_s e^{i2\pi ws+t} \\
&= \sum_{t=-\infty}^{\infty} \sum_{u-t=-\infty}^{\infty} x_u y_{t-u} e^{i2\pi wt} \text{ setting } s = t-u, t = u \\
&= \sum_{t=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} x_u y_{t-u} e^{i2\pi wt} \\
&= Z(\omega)
\end{aligned}
$$

Note that we can change this variables as such since this (1) these are infinite series with domain of $(-\infty, \infty)$ and so each variable achieves every possible Integer (2) the variables of one summation remains constant during the computation of the other.

b) By setting the $c_0 = 1$ and $c_0 = -1$ we notice that that $\Gamma(\omega) = 1 - e^{2\pi i\omega u}$,which leads us to find in that:

$$
\begin{aligned}
|\Gamma(\omega)|^2 &= |1 - e^{2\pi i\omega u}|^2 = |1 - e^{2\pi i\omega u}||1 - e^{-2\pi i\omega u}| \\
&= 2 - 2\cos(2\pi\omega)
\end{aligned}
$$

Thus removing such a trend from the data, removed lower frequencies in the data and highlight the higher frequencies. This was an evident change after taking the first difference in the data that the peak at frequency 0 was removed and the decaying trend had be more constant, with more emphasis on the lower frequencies.

c) By setting the gamma function given these parameters we get the weighted average of different possible lags, and so multiplying the absolute value of this by $\{x_t\}$ will remove or change the effect on the time series of the lags. Thus, this would detrend the data given the chosen weights and value of p, and will lead to more clear depiction of the data after reweighing the effects of the values of points that have a lag of p or less .

4 Question 4

a) First, we know that $e^{i\theta^k} = e^{ki\theta}$ by De Moivre's formula:

$$\cos(k\theta) + i\sin(k\theta) = (\cos(\theta) + i\sin(\theta))^k$$

And, so ,it is obvious and so we have that:

$$e^{ki\theta} = \cos(k\theta) + i\sin(k\theta) = (\cos(\theta) + i\sin(\theta))^k = e^{i\theta^k}$$

And since this holds for each term for each term in the bellow series, we have that:

$$\sum_{k=0}^{n-1} e^{(i\theta)^k} = \sum_{k=0}^{n-1} e^{i\theta k}$$

Then we can prove the next equality by induction. First we prove the base case, where $n = 1$. This is true, since both sides then equal 1. We commit to the induction hypothesis that for n = t:

$$\sum_{k=0}^{t-1} e^{ik\theta} = \frac{1 - e^{ti\theta}}{1 - e^{i\theta}}$$

Then we prove for the case of $n = t + 1$:

$$\begin{aligned}\sum_{k=0}^{t} e^{ik\theta} &= \sum_{k=0}^{t-1} e^{ik\theta} + e^{ti\theta} \\ &= \frac{1 - e^{ti\theta}}{1 - e^{i\theta}} + e^{ti\theta} \\ &= \frac{1 + e^{i\theta t}e^{i\theta}}{1 - e^{i\theta}} \\ &= \frac{1 + e^{(t+1)i\theta}}{1 - e^{i\theta}}\end{aligned}$$

This proves the equivalence of the two equations. Next we use this equivalence to find new representations of the sums of the sinusoidal functions:

Lemma 1 :
$e^{ik\theta} + e^{-ik\theta}$ $= \cos(k\theta) + i\sin(k\theta) + \cos(-k\theta) + i\sin(-k\theta)$ $= \cos(k\theta) + i\sin(k\theta) + \cos(k\theta) - i\sin(k\theta)$ $= 2\cos(k\theta)$
Lemma 2 :
$\frac{e^{ik\theta} - e^{-ik\theta}}{i} = -i(e^{ik\theta} - e^{-ik\theta})$ $= -i[\cos(k\theta) + i\sin(k\theta) - \cos(-k\theta) - i\sin(-k\theta)]$ $= -i[\cos(k\theta) + i\sin(k\theta) - \cos(k\theta) + i\sin(k\theta)]$ $= 2\sin(k\theta)$

$$
\begin{aligned}
\sum_{k=0}^{n-1} \cos(k\theta) &= \frac{1}{2} \sum_{k=0}^{n-1} e^{ki\theta} + e^{-ki\theta} \\
&= \frac{1}{2}\left(\frac{1-e^{ni\theta}}{1-e^{i\theta}} + \frac{1-e^{-ni\theta}}{1-e^{-i\theta}}\right) \\
&= \frac{1-e^{in\theta}-e^{i\theta}+e^{i\theta(n-1)}+1-e^{-in\theta}-e^{-i\theta}+e^{-i\theta(n-1)}}{2(2-(e^{i\theta}+e^{-i\theta}))} \\
&= \frac{2-(e^{in\theta}+e^{-in\theta})-(e^{i\theta}+e^{-i\theta})+(e^{i\theta(n-1)}+e^{-i\theta(n-1)})}{4(1-\cos(\theta))} \\
&= \frac{1-\cos(n\theta)-\cos(\theta)+cos((n-1)\theta)}{2(1-\cos(\theta))} \\
&= \frac{1-\cos(n\theta)-\cos(\theta)+\cos((n-1)\theta)}{2(1-\cos(\theta))} \\
&= \frac{\cos((n-1)\theta)-\cos(n\theta)}{2-2\cos(\theta))} + \frac{1}{2}
\end{aligned}
$$

And, (similaryly)

$$
\begin{aligned}
\sum_{k=0}^{n-1} \sin(k\theta) &= \sum_{k=0}^{n-1} \frac{e^{ki\theta}-e^{-ki\theta}}{2i} \\
&= \frac{1}{2i}\left(\frac{1-e^{ni\theta}}{1-e^{i\theta}} - \frac{1-e^{-ni\theta}}{1-e^{-i\theta}}\right) \\
&= \frac{1}{2i}\left(\frac{1-e^{ni\theta}}{1-e^{i\theta}} + \frac{e^{-ni\theta}-1}{1-e^{-i\theta}}\right) \\
&= \frac{1}{2i}\left(\frac{1-e^{in\theta}-e^{-i\theta}+e^{i\theta(n-1)}-1+e^{-in\theta}+e^{i\theta}-e^{-i\theta(n-1)}}{2-2cos(\theta)}\right) \\
&= \frac{1}{2i}\left(\frac{-(e^{in\theta}-e^{-in\theta})+(e^{i\theta}-e^{-i\theta})+(e^{i\theta(n-1)}-e^{-i\theta(n-1)})}{2-2cos(\theta)}\right) \\
&= \frac{1}{2}\left(\frac{-2\sin(n\theta)+2\sin(\theta)+2\sin(\theta(n-1))}{2-2\cos(\theta)}\right) \\
&= \frac{\sin(\theta)-\sin(n\theta)+\sin(\theta(n-1))}{2-2\cos(\theta)}
\end{aligned}
$$

(Note that the Euler's formula was used to attain the first equality in both formulas) b)

$$
\begin{aligned}
x_s &= \frac{1}{n}\sum_{k}^{n-1}\sum_{t}^{n} x_t e^{i2\pi\frac{k}{n}(t-s)} \\
&= \frac{1}{n}\sum_{k}^{n-1}\sum_{t=s}^{n} x_t e^{i2\pi\frac{k}{n}(t-s)} + \frac{1}{n}\sum_{k}^{n-1}\sum_{t\neq s}^{n} x_t e^{i2\pi\frac{k}{n}(t-s)} \\
&= \frac{n}{n}x_s + \frac{1}{n}\sum_{t\neq s}^{n} x_t \sum_{k}^{n-1} e^{i2\pi\frac{k}{n}(t-s)} \\
&= x_s + \frac{1}{n}\sum_{t\neq s}^{n} x_t \frac{1-e^{i2\pi(s-t)}}{1-e^{i2\pi\frac{(s-t)}{n}}} \\
&= x_s
\end{aligned}
$$

Note that the final equality is due to the fact that $2\pi(s-t)$ is multiple of 2π, and so $e^{i2\pi(s-t)} = 1$.

5 Appnedix

Figure 1

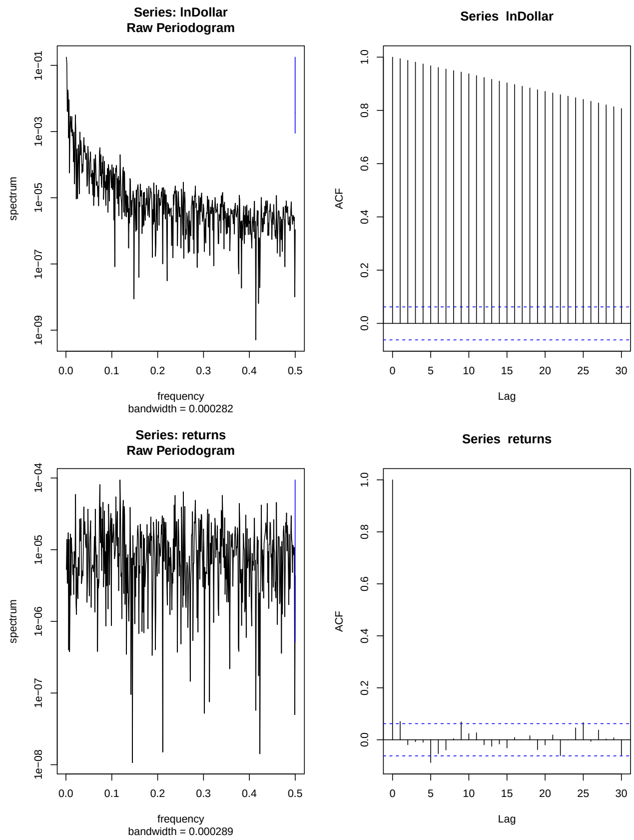

Figure 3

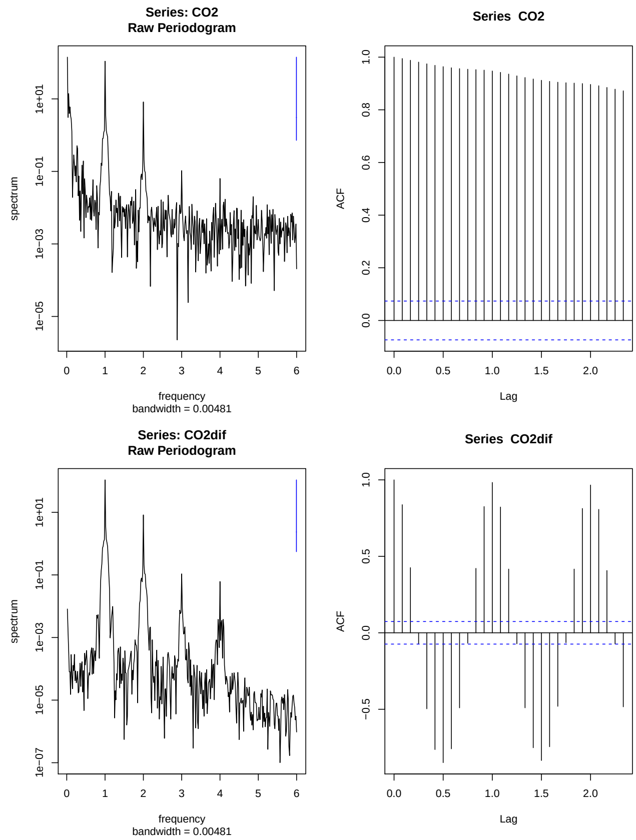

Figure 2