Introductory Real Analysis: Interesting Problems

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1 Limits

1.1 Evaluating a Classical Limit

Evaluate $\lim_{n\to\infty} 1 - \frac{n}{n^2-1}$ *using arithmetic of limits.*

We recall the "Squeeze Theorem", which states: Let I be an interval having the point a as a limit point. Let f, g and h be functions defined on I, except possibly at a itself. Suppose that for every $x \in I$ not equal to a, we have:

$$g(x) \leq f(x) \leq h(x) \text{ and } \lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

then, $\lim_{x\to a} f(x) = L$.

Thus, let $\xi(n) = 1$ and $f(n) = \frac{n}{n^2-1}$, then $\lim_{n\to\infty}(1 - \frac{n}{n^2-1}) = \lim_{n\to\infty}(\xi(n)) - \lim_{n\to\infty} f(n)$. We now note if $\aleph = (n-1)(n+1)$, then $f(n) < f(n) + \frac{1}{\aleph}$ and $f(n) > f(n) - \frac{1}{\aleph}$ $(\forall\ n > 1)$. Therefore if we have $h(x) = f(n) + \frac{1}{\aleph} = \frac{1}{n-1}$ and $g(x) = f(n) - \frac{1}{\aleph} = \frac{1}{n+1}$, then by the squeeze theorem $\lim_{n\to\infty} f(n) = 0$ since $\lim_{n\to\infty} g(n) = \lim_{n\to\infty} h(n)$. Thus, we may conclude:

$$\lim_{n\to\infty}\left(1 - \frac{n}{n^2-1}\right) = \lim_{n\to\infty} \xi(n) = 1$$

Another approach to this problem would be as follows. We recall $\lim_{n\to\infty} \frac{1}{n} = 0$. Then, by the standard arithmetic of limits:

$$\lim_{n\to\infty}(1 - \frac{n}{n^2-1}) = \lim_{n\to\infty}(1) - \lim_{n\to\infty}\frac{n}{n^2-1} = 1 - \frac{\lim_{n\to\infty} n\frac{1}{n^2}}{\lim_{n\to\infty}(n^2-1)\frac{1}{n^2}} = 1 - \frac{\lim_{n\to\infty}(\frac{1}{n})}{\lim_{n\to\infty}(1-\frac{1}{n^2})}$$

$$= 1 - \frac{0}{\lim_{n\to\infty}(1) - \lim_{n\to\infty}(\frac{1}{n})\lim_{n\to\infty}(\frac{1}{n})} = 1 - \frac{0}{1-(0)(0)} = 1$$

1.2 Convergent Sequences

Define a real sequence (x_n) *by* $x_1 = 0$ *and* $x_{n+1} = \frac{1}{4(1-x_n)}$ *for* $n \geq 1$. *Show that* (x_n) *is convergent, and find the limit.*

Proof. Let us begin by noting the first 11 terms of x_n as $x_n = \{0, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{5}{12}, \frac{3}{7}, \frac{7}{16}, \frac{4}{9}, \frac{9}{10}, \frac{5}{11}, \dots\}$. Thus, a quick glance at this list looks like there might exist a direct formula for all x_n's not dependant on x_n. We claim that:

$$\forall n = 2k+1, k \in \mathbb{N}, x_{2k+1} = \frac{k}{2k+1}, \text{ and } \forall n = 2k, k \in \mathbb{N}, x_{2k} = \frac{2k-1}{4k}, \text{and } x_1 = 0$$

Our formulas cover $x_n \forall n$. Therefore, our plan is to perform induction on both formulas, to prove correct description, then compute the limits.

For $n = 2k+1$, we note $x_{2(1)+1} = \frac{1}{3} = \frac{1}{2(1)+1}$. Next, we assume for $n = 2k+1, x_{2k+1} = \frac{k}{2k+1}$, then for $n = 2(k+1)+1 = 2k+3 = n+2$,

$$x_{n+2} = \frac{1}{4(1-\frac{1}{4(1-x_n)})} = \frac{1}{4(1-\frac{1}{4(1-\frac{k}{2k+1})})} = \frac{1}{4-(\frac{1}{(\frac{k+1}{2k+1})})} = \frac{1}{(\frac{4k+4-2k-1}{k+1})} = \frac{(k+1)}{2(k+1)+1}$$

Which is our formula for $x_{2(k+1)+1}$.

Now, for $n = 2k+1$, we note $x_{2(1)} = \frac{1}{4} = \frac{2(1)-1}{4(1)}$. Next, we assume for $n = 2k, x_{2k} = \frac{2k-1}{4k}$, then for $n = 2(k+1) = 2k+2 = n+2$,

$$x_{n+2} = \frac{1}{4(1 - \frac{1}{4(1-x_n)})} = \frac{1}{4(1 - \frac{1}{4(1-\frac{2k-1}{4k})})} = \frac{1}{4 - (\frac{4k}{2k+1})} = \frac{1}{(\frac{4k+4}{2k+1})} = \frac{1}{(\frac{4(k+1)}{2(k+1)-1})} = \frac{2(k+1)-1}{4(k+1)}$$

Which is our formula for $x_{2(k+1)}$.

Now, we want to show $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{k}{2k+1} = \lim_{n\to\infty} \frac{2k-1}{4k} (< \infty)$. We claim $L = \frac{1}{2}$ is the limit of all these sequences. To check, given $\epsilon > 0$, let $N > \frac{1}{4}(\frac{1}{\epsilon} - 2)$. Thus $\forall k > N$,

$$|\frac{k}{2k+1} - \frac{1}{2}| = \frac{1}{2} - \frac{k}{2k+1} \leq \frac{1}{2} - \frac{N}{2N+1} < \frac{1}{2} - \frac{\frac{1}{4}(\frac{1}{\epsilon} - 2)}{(\frac{1}{\epsilon} - 1) + 1} = \frac{1}{2} - \frac{\frac{1-2\epsilon}{4\epsilon}}{\frac{1}{2\epsilon}} = \frac{1}{2} - (\frac{1}{2} - \epsilon) = \epsilon$$

since if $0 < \frac{x}{y} < 1$, then $\frac{x-1}{y-1} < \frac{x}{y}$. Now similarly but with $N > \frac{1}{4\epsilon}$,

$$|\frac{2k-1}{4k} - \frac{1}{2}| = \frac{1}{2} - \frac{2k-1}{4k} \leq \frac{1}{2} - \frac{2N-1}{4N} < \frac{1}{2} - \frac{2(\frac{1}{4\epsilon}) - 1}{4(\frac{1}{4\epsilon})} = \frac{1}{2} - (\frac{1}{2} - \epsilon) = \epsilon$$

Therefore, $x_n \to \frac{1}{2}$ as $n \to \infty$.

$\square$

1.3 A Nice Property of Limits

Let (x_n) be a sequence of real numbers that converges to x, and let $a, b \in \mathbb{R}$.

1. *Show that, if $x_n \leq b$ for every $n \in \mathbb{N}$, then $x \leq b$. What can you conclude if $x_n < b$ for every $n \in \mathbb{N}$?*
2. *Deduce from (1) that, if $x_n \geq a$ for every $n \in \mathbb{N}$, then $x \geq a$.*

1. *Proof.* *** (Insufficient proof for 1., need to approach from different perspective, need to better define the leap from $\lim_{n\to\infty}(x_n - b_n) \leq 0 \implies x \leq b$).

   Let us define $b_n = b \ \forall n$. Then $\lim_{n\to\infty} x_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty}(x_n - b_n) \leq 0$ since $x_n \leq b_n \forall n$. Therefore, $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} b_n \equiv x \leq b$. One might think that the added information that $x_n < b \forall n \implies x < b$. However, consider the case where $x_n = 1 - \frac{1}{n}$ and $b = 1$. We note this sequence and b satisfy our conditions $x_n < b \forall n$. However, $\lim_{n\to\infty} x_n = 1 = b$. Therefore, we must conclude that since $x_n < b$ is stronger than $x_n \leq b$, but doesn't add any additional information for cases such as the one we just provided, that both conditions are sufficiently equivalent for our conclusion about $x \leq b$.

   $\square$

2. Let us consider $y_n = -x_n$, where $\lim_{n\to\infty} y_n = y$ and $a = -b$. We note if $x_n \leq b \ \forall n$, then $-y_n \leq -a \implies y_n \geq a \ \forall n$. Thus, by the completeness of $\mathbb{R}$, we may conclude that $\forall$ series $y_n \to y$ as $n \to \infty$ where $y_n \geq a \ \forall n \implies y \geq a$ from our findings in (1).

2 Sets

2.1 Set Addition

Let S and T be nonempty subsets of $\mathbb{R}$ that are both bounded above. Prove that the set $S + T = \{s + t | s \in S, t \in T\}$ has $\sup(S + T) = \sup S + \sup T$.

Proof. Let us consider 4 cases:

- First, assume $s_0 = \sup(S), s_0 \in S$ and $t_0 = \sup(S), t_0 \in T$. Therefore, given t, and since $S + T \in \mathbb{R}$, the maximum value $S + T$ could take would be $s_o + t$, and similarly for a case given s. Therefore, $\max(s + t) = s_0 + t_0 \equiv \sup(S) + \sup(T)$.
- Second, let us consider the case where both $\sup(S)$ $\sup(T) \notin S$ and T respectively. Therefore, $\lim_{s \to \sup(S)} s = \sup(S) = s_0$, and similarly for t. Therefore, given t, and since $S + T \in \mathbb{R}$, the $\sup(S + t)$ would be $\lim_{s \to \sup(S)} s + t = s_0 + t$, and similarly for a case given s. Thus, $\sup(S + T) = s_0 + t_0 \equiv \sup(S) + \sup(T)$.
- The third and fourth cases are symmetric, therefore, WLOG, assume $\lim_{s \to \sup(S)} s = \sup(S) = s_0$, $s_0 \notin S$, but $\sup(T) = t_0$ and $t_0 \in T$. We combine the previous two cases together to find the supremum of $S+T$ given one variable being constant, which yields $\sup(S+T) = s_0 + t_0 = \sup(S) + \sup(T)$.

Since all possible cases give the case where $\sup(S + T) = \sup(S) + \sup(T)$, we can conclude what we wanted to show. □

2.2 Proof of the Existence of a Convergence Sequence of elements of a set to its Infimum

Prove that, if S is a nonempty subset of $\mathbb{R}$ that is bounded below, then there is a monotone decreasing sequence of elements of S that converges to $\inf S$.

Proof. Let us define $B_r(x) = \{x \in \mathbb{R} : x < r\}$. Therefore, we have two cases:

- $\exists t > 0$ s.t. $\forall r < t, B_r(\inf(S)) = \inf(S)$
- $\nexists t > 0$ s.t. $\forall r < t, B_r(\inf(S)) = \inf(S)$

  In the first case, our condition implies $\inf(S) \in S$. To prove this assume $\inf(S) \notin S$. Therefore, $\forall \epsilon > 0, \exists s \in S$ s.t. $|s - \inf(S)| < \epsilon$. This is a contradiction since our first case tells us for some $t > 0 \nexists$ any points other than possibly $\inf(S)$ in S and in an open ball around $\inf(S)$. Thus, if the first condition holds, we define our monotone decreasing sequence as the constant sequence $x_n = \inf(S)$.

  For the second case, if $\inf(S) \in S$, we let $x_n = \inf(S)\ \forall n$. Otherwise, take $r_1 = t$ where t is the first such radius where $\forall x \in B_{r_1}(\inf(S))$ to the right of $\inf(S)$ $x \in S$, unless $S = (\inf(S), \infty)$ in which we take $r_1 = 1$. For example, if $S = \cup_{m=0}^{\infty}(m, m + \frac{1}{2}]$, then $r_1 = \frac{1}{2}$. With r_1, we take $r_{n+1} = \frac{r_n}{2}\ \forall n \in \mathbb{N}$ where $r_1 = t$ as defined before. Then, we take the sequence $y_n = \sup(B_{r_n}(\inf(S)))$ which must converge to $\inf(S)$ since our ball's radius is shrinking to 0 around $\inf(S)$, and $y_n \in S\ \forall n$, thus y_n satisfies the conditions of a sequence we were looking for.

□

3 Convergent Sums

3.1 Proving Absolute Convergence

Suppose that (a_n) *is a sequence of positive real numbers with* $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = s < 1$. *Show that* $\sum a_n$ *is absolutely convergent. Deduce that* $\sum \frac{x^n}{n!}$ *converges for every* $x \in \mathbb{R}$.

Proof. Let us define $r = \frac{s+1}{2}$. By the completeness of $\mathbb{R}$ and how $s < 1 \implies s < r < 1$. It follows that if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = s < 1$, then by definition, $\forall \epsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $|\frac{a_{n+1}}{a_n} - s| < \epsilon \ \forall n > N \implies |a_{n+1}| < r|a_n| \ \forall n > N, N \in \mathbb{N}$. Therefore, $|a_{n+i}| < r|a_{n+i-1}| < \cdots < r^i|a_n| \ \forall i > 0, i \in \mathbb{N}$. As such,

$$\sum_{i=N+1}^{\infty} |a_i| = \sum_{i=1}^{\infty} |a_{N+i}| < \sum_{i=1}^{\infty} r^i |a_{N+1}| = |a_{N+1}| \sum_{i=1}^{\infty} r^i = |a_{N+1}| \frac{r}{1-r} < \infty \text{ since } r < 1$$

Therefore, $\sum a_n$ is absolutely convergent.

$\square$

Proof. As for showing $\sum \frac{x^n}{n!} = \sum a_n$ converges for every $x \in \mathbb{R}$. We prove that $\forall n > k$, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = s < 1$ since $\forall n > k$

$$\frac{k^n}{(n)!} = (\frac{k^k}{k!})(\frac{k}{k+1}) \ldots (\frac{k}{n}) < (\frac{k^k}{k!})(\frac{k}{n}) < (\frac{k^k}{k!}) = a_k$$

and $\forall m \geq n$, a_{m+1} satisfies:

$$\frac{k^{m+1}}{(m+1)!} = (\frac{k}{m+1})(\frac{k^m}{m!}) < (\frac{k^m}{m!}) = a_m \leq (\frac{k^k}{k!}) = a_k$$

This $\implies \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = s < 1$. The fact that $\sum a_n$ converges now follows from our previous findings. $\square$

4 Continuity

Let us first recall the Intermediate Value Theorem to be used heavily in the next few questions:

Theorem. 4.1: The Intermediate Value Theorem on $\mathbb{R}$

If f is a real valued continuous function on $[a, b]$ s.t. $f(a) < f(b)$, then $\forall y \in [f(a), f(b)]$, $\exists x \in [a, b]$ s.t. $f(x) = y$.

4.1 Proving Uniform Continuity Example

Show that the function $f : [0, \infty) \to \mathbb{R}, f(x) = x^2$ *is not uniformly continuous.*

Proof. We recall that a function is uniformly continuous from $S \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $||f(\boldsymbol{x}) - f(\boldsymbol{a})|| < \epsilon$ when $||\boldsymbol{x} - \boldsymbol{a}|| < \delta$, $\boldsymbol{x}, \boldsymbol{a} \in S$. Therefore, given $\epsilon = 1$ suppose there exists

$\delta > 0$ s.t. the δ satisfies our definition of uniform continuity. However, if we consider the sequence $x_n = n, a_n = n + \frac{1}{n}$:

$$|f(x_n) - f(a_n)| = (n + \frac{1}{n})^2 - n^2 = 2 + \frac{1}{n^2}$$

Therefore, $\forall\delta > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$, which $\implies$ $|x_n - a_n| = |n - n - \frac{1}{n}| = \frac{1}{n} < \delta$, but $|f(x_n) - f(a_n)| > 2 > \epsilon$. Thus, we can now conclude f on the domain $S = [0, \infty)$ is not uniformly continuous.

□

4.2 Continuous and Differentiable Proof

Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$ and differentiable on $(1, \infty)$ with bounded derivative. Show that f is uniformly continuous. (HINT: split $[0, \infty)$ into two pieces, on each of which f is uniformly continuous, then explain why this implies that f is uniformly continuous on the whole of $[0, \infty)$.)

Proof. We split the domain of $[0, \infty)$ into two sets:

1. We first consider f under the interval $[0, 1]$, which must be uniformly continuous since the domain is compact and f is continuous (Theorem 5.5.9)
2. We now consider f under the domain $(1, \infty)$. Since f is continuous and has a bounded derivative under this domain, we know $\exists c \in (1, \infty)$ s.t. $f'(c) \geq \frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|}$ $\forall x_1, x_2 \in (1, \infty) \implies$ $|f(x_2) - f(x_1)| \leq f'(c)|x_2 - x_1|$ and hence f under $(1, \infty)$ is Lipschitz. We now recall that all Lipschitz functions are also uniformly continuous (Prop. 5.5.4) and hence f under $(1, \infty)$ is uniformly continuous.

Since we have shown uniform continuity under the two domains, we must show uniform continuity under the union. Thus, choose δ_1, δ_2 s.t. $\forall y \in (1, \infty), x \in [0, 1]$, having $|1 - y| < \delta_1 \implies |f(1) - f(x)| < \frac{\epsilon}{2}$ and $|x - 1| < \delta_2 \implies |f(y) - f(1)| < \frac{\epsilon}{2}$ and hence $|f(y) - f(x)| \leq |f(1) - f(x)| + |f(y) - f(1)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$ when $\delta = \max(\delta_1, \delta_2)$. Therefore, all of $[0, \infty)$ is uniformly continuous. □

4.3 Proving Uniform Continuity

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(x+1) = f(x) \forall x \in \mathbb{R}$. Show that f is uniformly continuous.

Proof. We first note under the domain $[0, 1]$, f will be uniformly continuous since under compact domains, a continuous function is uniformly continuous (Theorem 5.5.9). Thus, $\forall x, y \in \mathbb{R}$, WLOG $y > x$ we have $|f(y) - f(x)| \leq |f(y) - f(y-1)| + |f(y-1) - f(y-2)| + \cdots + |f(y-n) - f(x)|$ where $y - n \leq x + 1$, and hence $|f(y) - f(x)| \leq |f(y-n) - f(x)|$ since $|f(z) - f(z-1)| = 0$ by definition of f.

Thus, we now note the uniform continuity we established under $[0, 1]$ may be generalized to all of $\mathbb{R}$ since once we note any $|f(y) - f(x)| \leq |f(y-n) - f(x)|$ where $y - n \leq x + 1$, we may then push both $y - n$ and x down to elements of $[0, 2]$. since $|f(y-n) - f(x)| = f(y-n-k) - f(x-k)$ $\forall k \in \mathbb{N}$ and requiring . Thus, all that needs to be done is to extend our domain of uniform continuity by 1. Quite trivially, we note our findings of why f under $[1 - \epsilon, 2 - \epsilon]$ $(\epsilon \in [0, 1])$ or $[1, 2]$ is also uniformly continuous (by compactness and continuity), and hence f under $[0, 2]$ is uniformly continuous. Thus, if we require $x-k \in [0,1]$ we have all the tools to now conclude that $\mathbb{R}$ is uniformly continuous since $\forall x, y \in \mathbb{R}\ |f(y) - f(x)| \equiv |f(z_2) - f(z_1)|, z_1, z_2 \in [0,2]$ and $z_2 \le z_1 + 1$.

□

4.4 Surjectivity

Suppose that $f : \mathbb{R} \to [0,\infty)$ is continuous, unbounded, and satisfies $f(0) = 0$. Show that $f(\mathbb{R}) = [0,\infty)$ (that is, the range of f is all of $[0,\infty)$).

Proof. $\forall M > 0$, by Intermdiate Value Theorem (IVT), if $f(m) = M$, then $\forall y \in [0,M]\ \exists x$ s.t. $f(x) = y$. By unboundness, $\forall l,\ \exists L_1 \in f(\mathbb{R}) = [0,\infty)$ s.t. $B_l(0) \cap \{L\} = \emptyset$. Therefore, $\forall n \in \mathbb{N},\ \exists L_n > n$ s.t. $B_n(0) \cap \{L_n\} = \emptyset \implies \{0, L_n\} \subseteq \text{Range}(f)$. Thus, by IVT, $[0, L_n] \subseteq \text{Range}(f)$, and since $\lim_{n\to\infty} L_n = +\infty$, it $\implies$ Range(f)=$[0,\infty)$ □

4.5 Proving the Existence of a Point in a Functions Range

Show that there is some $x \in \mathbb{R}$ such that $\sqrt{x} + \sqrt{\cos(\sin(x))} = 2$.

Proof. With consideration of IVT, if we can (1) Show $f(x) = \sqrt{x} + \sqrt{\cos(\sin(x))} = 2$ is continuous on some S, and (2) Show $\exists a, b \in S$ s.t. $f(a) < 2 < f(b)$, we will have proven $\exists x$ s.t. $f(x) = 2$. To show (1), let us recall the following two theorems: If f, g are continuous real-valued functions under the common domain S, then $f(g(x))$ is also continuous. The second theorem is that under the same conditions, $f + g$ is also continuous.

We quickly prove $\sqrt{x}$ is continuous under $[0,\infty)$ since if we require $\delta = \min(1,\ \varepsilon(\sqrt{a+1} + \sqrt{a}))$, then having $|x-a| < \delta \implies |\sqrt{x} - \sqrt{a}| < \epsilon$ since $|\sqrt{x} - \sqrt{a}| < \epsilon \iff |x-a| < \epsilon \cdot |\sqrt{x} + \sqrt{a}|$, and also $\sqrt{x}$ is not defined on $x < 0$ (under $\mathbb{R}$) and hence only the right-hand limit matters at 0 and $\sqrt{0} := 0$. Therefore, $\sqrt{x}$ is continuous under $[0,\infty)$.

Therefore, letting $f_1(x) = x,\ f_2(x) = \sin(x),\ f_3(x) = \cos(x),\ g(x) = \sqrt{x}$, and knowing all these functions are continuous under $[0,\infty]$, we now know $h(x) = g(f_1(x)) + g(f_3(f_2(x)))$ is also continous under $[0,\infty)$. Therefore, since $h(0) = \sqrt{0} + \sqrt{1} = 1 < 2$ and $h(2\pi) = \sqrt{2\pi} + 1 > 2$, by IVT we may conclude $\exists\ x \in (0,\infty)$ s.t. $h(x) = 2$.

□