Partial Differential Equations: Assigned Problems

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September 29, 2017

1 Assignment 1

Question 1

Exercise. 1.1:

Consider the second order nonlinear PDE appearing in the theory of minimal surfaces

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0 \tag{1}$$

Assume that u is radially symmetric, i.e. u depends only on the radial coordinate $r = \sqrt{x^2 + y^2}$, thus write $u(x, y) = h(r)$.

(a) Show that the function h satisfies:

$$rh'' + h'(1 + (h')^2) = 0$$

(b) Solve equation (1) and write its general solution.

(a) We first find (via the Chain Rule) that:

$$\begin{aligned}
u_x &= (h')(x/r) \\
u_y &= (h')(y/r) \\
u_{xx} &= (h'')(x^2/r^2) + (h')(1/r - x^2/r^3) \\
u_{yy} &= (h'')(y^2/r^2) + (h')(1/r - y^2/r^3) \\
u_{xy} &= (h'')(yx/r^2) - (h')(xy/r^3)
\end{aligned}$$

And therefore, plugging the above into Eq. (1) and multiplying things out, we get:

$$\begin{aligned}
Eq(1) &= h''r + 2h'/r - h'/r + 2h'h'h''x^2y^2/r^4 + rh'h'h' - 2x^2y^2 - 2x^2y^2/r^4 h'h'h'' + 2x^2y^2/r^5 h'h'h' \\
&= h''r + h' + h'h'h' \\
&= rh'' + h'(1 + (h')^2)
\end{aligned}$$

(b) Let us set $h' = f$, and hence our second order ODE becomes first order in the form of:

$$rf' + f(1 + f^2) = 0$$

For which has the solution:

$$f = \pm \frac{ie^{c_1}}{\sqrt{e^{2c_1} - r^2}}$$

And thus integrating:

$$h(r) = \int f = c_2 \pm ie^{c_1} \tan^{-1}\left(\frac{r}{\sqrt{e^{c+1} - r^2}}\right)$$

Question 2

Exercise. 1.2:

Let $u(x,t)$ in $C^2(\mathbb{R} \times [0,\infty))$ solve the wave equation:

$$u_{tt} - u_{xx} = 0$$

with the initial condition:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

with compact support. (This means that f and g are identically 0 outside an interval $(-R, R)$.) Let $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ and $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$ be the kinetic and potential energy.

(a) Prove that the total energy $E(t) = k(t) + p(t)$ is conserved.

(b) Prove that in the limit of $t \to \infty$, $k(t) = p(t)$. This property is referred to as "Equipartition of energy".

(a) *Proof.* We recall that conservation of energy means that $\partial_t E(t) = 0$. Therefore, we investigate the E's derivative w.r.t. t as follows:

$$\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left(u_t^2 + u_x^2\right)dx\right) \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(u_t^2 + u_x^2 dx\right) && \text{because of } * \text{ (below)} \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\left(2u_t u_{tt} + 2u_x u_{xt}\right)dx \\
&= \int_{-\infty}^{\infty}\left(u_t u_{tt} + u_x u_{xt}\right)dx
\end{aligned}$$

$*$: $d/dt \int = \int \partial_t$ because u is identically zero for $|x|$ large, and is C^2.

We now modify the $\int_{-\infty}^{\infty} u_x u_{xt} dx$ term by integrating by parts:

$$\begin{aligned}
\int_{-\infty}^{\infty} u_x u_{tx} dx &= u_t u_x \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_{xx} u_t dx \\
&= -\int_{-\infty}^{\infty} u_{xx} u_t dx && \text{since } u \equiv 0 \text{ for } |x| >> 0 \\
&= -\int_{-\infty}^{\infty} u_{tt} u_t dx && \text{since } u_{tt} = u_{xx}
\end{aligned}$$

And hence it is clear that:

$$
\begin{aligned}
\frac{dE}{dt} &= \int_{-\infty}^{\infty} (u_t u_{tt} + u_x u_{xt})\, dx \\
&= \int_{-\infty}^{\infty} (u_t u_{tt} - u_t u_{tt})\, dx \\
&= 0
\end{aligned}
$$

□

(b) *Proof.* To solve this problem, we recall the following Lemma:

Lemma. 1.1: d'Alembert's Formula

The one-dimensional Solution to the Wave Equation defined above is given by:

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi) d\xi$$

From this, we calculate explicitly u_t and u_x:

$$
\begin{aligned}
u_t &= \frac{1}{2}\Big[f'(x+t) - f'(x-t)\Big] + \frac{1}{2}\Big[g'(x+t) + g'(x-t)\Big] \\
u_x &= \frac{1}{2}\Big[f'(x+t) + f'(x-t)\Big] + \frac{1}{2}\Big[g'(x+t) - g'(x-t)\Big]
\end{aligned}
$$

And since $\int_{-\infty}^{\infty} u_x^2 dx = \int_{-\infty}^{\infty} u_t^2 dx \iff 0 = \int_{-\infty}^{\infty} (u_x^2 - u_t^2) dx$, we'll take a look at $u_x^2 - u_t^2$:

$$
\begin{aligned}
u_x^2 - u_t^2 &= (u_x - u_t)(u_x + u_t) \\
&= (f'(x-t) - g'(x-t))(f'(x+t) + g'(x+t)) \\
&= f'(x-t)f'(x+t) - g'(x-t)f'(x+t) + f'(x-t)g'(x+t) - g'(x-t)g'(x+t) \\
&= 0
\end{aligned}
$$

Where the last equality is since $(x+t) - (x-t) = 2t$, when $t > R \Rightarrow$ at least one of the following $(x-t)$ or $(x+t)$ will $\notin [-R, R]$, and since f, g are both identically zero outside the interval $[-R, R]$, we thus have every every expression in the above evaluating to zero since all terms are in the form of $h_1(x-t)h_2(x+t)$ where $h_1, h_2 = f$ or g. □

Question 3

Exercise. 1.3:

Prove the comparison principle for the diffusion equation:

$$u_t - u_{xx} = 0; \quad x \in (0, l), t > 0$$

(a) If u and v are two solutions and if $u \leq v$, at $t = 0$ and for $x = 0$ and $x = l$, then $u \leq v$ for all $t \geq 0$ and $0 \leq x \leq l$.

(b) The purpose of this question is to prove a more general comparison principle. Assume that $u_t - u_{xx} = f$, $v_t - v_{xx} = g$, $f \leq g$ and $u \leq v$, at $t = 0$ and for $x = 0$ and $x = l$. Consider $w = u - v$. For $\varepsilon > 0$, introduce the function:

$$W(x, t) = w(x, t) + \varepsilon x^2$$

Fix $T > 0$. Show that W has no interior maximum in the rectangle $[0, l] \times [0, T)$. Show that it cannot have a maximum a point (x, T), with $0 < x < l$. Prove that $u \leq v$ for all $t \geq 0$ and $0 \leq x \leq l$.

To solve the above exercise, we introduce and give a quick proof for the following Maximum Principle:

Lemma. 1.2: The Maximum Principle for the Heat Equation

If $u \in C^2(U_T) \cap C^1\left(\overline{U}_T\right)$ solves the linear heat equation, then:

$$\max_{(x,t)\in\overline{U}_T}(u) = \max_{(x,t)\in\Gamma_T}(u)$$

Where $U_T := U(0, T]$, $U \subset \mathbb{R}^n$, U open and bounded, and $\Gamma_T := \overline{U}_T \backslash U$.

Proof. □

(a) Let us define $w := u - v$. Since $D^\alpha h = D^\alpha u - D^\alpha v \ \forall |\alpha| \leq 2$, it must be that h satisfies $h_t = h_{xx}$ since u and v do. Now, since $w \leq 0 \ \forall (x, t) \in \{(x, t) \mid x = 0 \text{ or } x = l \text{ or } t = 0\}$, by the Maximum Principle, $w \leq 0 \ \forall (x, t) \in U_T = \{(x, t) \ |0 < x < l, 0 < t \leq T\}$. Now, since T was arbitrary, we may let $T \to \infty$, and hence $w \leq 0 \ \forall t \geq 0$, and since $w := u - v \Rightarrow u \leq v \ \forall (x, t) \in \overline{U}_T$.

(b) Let us introduce w and W as advised, and for the sake of contradiction, assume $(x_0, t_0) = \text{argmax}_{(x,t)\in\overline{U}_T}(W)$ and $(x_0, t_0) \in (0, l) \times (0, T)$. The most immediate observation we can see is that $W_t = w_t$ and $W_{xx} = w_{xx} + 2\epsilon$. Furthermore, since both u and v satisfy their respective heat equations, we must have $w_t - w_{xx} = f - g \leq 0$. And lastley, since $\text{argmax}_{(x,t)\in\overline{U}_T}(W) = (x_0, t_0)$ $\Rightarrow W_t(x_0, t_0) = 0$ and $W_{xx}(x_0, t_0) \leq 0$, and hence $W_t(x_0, t_0) - W_{xx}(x_0, t_0) \geq 0$. We can now come across the following contradiction:

$$W_t - W_{xx} = w_t - w_{xx} - 2\epsilon = (f - g) - 2\epsilon \leq -2\epsilon < 0$$

Which can not be so since we showed $W_t(x_0, t_0) - W_{xx}(x_0, t_0) \geq 0$. Thus, W cannot have a maximum in $(0, l) \times (0, T)$.

Let us now assume (again for contradiction) $(x_0, t_0) = \text{argmax}(W)$ and $(x_0, t_0) \in (0, l) \times \{T\}$. For this case, $W_t(x_0, T)$ will be it's derivative as $t \to T^-$, and hence $W_t(x_0, T) \geq 0$, and like before

$W_{xx}(x_0, T) \leq 0$. Therefore, at (x_0, T), it must be that $W_t(x_0, T) - W_{xx}(x_0, T) \geq 0$. However since $W_t = w_t$ and $W_{xx} = w_{xx} = f - g \leq 0$, we again have:

$$W_t - W_{xx} = w_t - w_{xx} - 2\epsilon = (f - g) - 2\epsilon \leq -2\epsilon < 0$$

Which again is a contradiction and hence W cannot have a maximum on $(0, l) \times \{T\}$.

To conclude that $w \leq \Rightarrow u \leq v \ \forall t \geq$, we make the following argument. Since T was arbitrary, like in (a), we choose larger and larger T, and hence the above arguments actually hold for $t \geq 0$. Furthermore, since we showed that W has no maximums in $(0, l) \times (0, T]$ (and since $\max(x) = l$), we can say that $W \leq \epsilon l^2$, and since $W := w + \epsilon x^2 \Rightarrow w \leq \epsilon(L^2 - x^2)$, and letting $\epsilon \to 0 \Rightarrow w \leq 0$, and hence $u \leq v \ \forall t \geq, 0 \leq x \leq l$.

Question 4

Exercise. 1.4:

Consider the first order equation:

$$u_t + tu_x = 0$$

(a) Find the characteristic curves in the (x, t) plane.

(b) Write the general solution.

(c) Solve equation (1) with initial condition $u(x, 0) = \sin(x)$. Explain why the solution is fully determined by the initial condition.

(a) If we consider a function, $w(X(t), t)$, then by the chain rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{dx}$$

Which we quickly recognize to be in a form useful for this question. Thus, setting $dw/dt = 0$ and $dX/dt = t$, we can ascertain the necessary characteristic curves will be $X = \frac{1}{2}t^2 + x_0$.

(b) Naturally, $du/dt = 0$ along the characteristic curves $dx/dt = t$. Therefore, u will be constant along the equation $x = t^2/2 + x_0$, and hence our general solution will be that of the family of functions which take the form:

$$u(x, t) = u(x_0, 0) = f(x_0) = f(x - t^2/2)$$

(c) From (3), we know that $u(x, t) = f(x - \frac{1}{2}t^2)$. Thus, if $u(x, 0) = \sin(x) \Rightarrow$ the unique way to write $u(x, t)$ is $\sin(x - \frac{1}{2}t^2)$ since this is the only equation which satisfies $u(x, 0) = \sin(x)$.

Question 5

Exercise. 1.5:

The purpose is to derive the formula for the inhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad x \in \mathbb{R} \tag{2}$$

with initial conditions:

$$u(x,0) = 0; \quad u_t(x,0) = 0 \tag{3}$$

(a) Let:

$$v = u_t + cu_x$$

Show that v satisfies the equation:

$$v_t - cv_x = f$$

(b) Write the corresponding initial conditions for $u(x,0)$ and $v(x,0)$.

(c) Solve equation $v = u_t + cu_x$ for u in terms of v.

(d) Solve equation $v_t - cv_x = f$ for v in terms of f.

(e) Substitute part (d) into part (c) to find the solution u of (2)-(3). Play with the double integral to identify your answer to:

$$u(x,t) = \int\int_{\Delta} f(y,s)dyds$$

where Δ is the characteristic triangle (make a picture).

(a) We compute:

$$v_t = u_{tt} + u_{xt}, \quad v_x = u_{tx} + cu_{xx}$$

Therefore:

$$\begin{aligned} v_t - cv_x &= u_{tt} + cu_{xt} - c(u_{tx} + cu_{xx}) \\ &= (u_{tt} - c^2 u_{xx}) + cu_{ct} - cu_{tx} \\ &= f && \text{since } u_{tt} - c^2 u_{xx} = f \end{aligned}$$

(b) Since v is a classical first order PDE, we only need one initial condition for v, namely:

$$v(x,0) = u_t(x,0) + cu_x(x,0) = 0 + c(0') = 0$$

and u has the same conditions as before.

(c) Solving this equation as we did in Question 4 involves introducing the equations $dx/dt = c$, and $du/dt = v$. Thus, the characteristics satisfy $x = ct + x_0$, and $u = \int vdt$. Thus, $u = h_1(x - ct) = \int vdt$ (I.e., the bounds of integration will be in terms of $(x - ct)$).

(d) Also solving this equation as we did in Question 4 involves introducing the equations $dx/dt = c$, and $dv/dt = f$. Thus, the characteristics satisfy $x = ct - x_0$, and $v = \int fdt$. Thus, $v = h_2(x + ct) = \int fdt$ (I.e., the bounds of integration will be in terms of $(x + ct)$).

(e) Firstly, we note that our "charactersitic traingles" will be defined by the triangle construed the interor of the lines $x_0 = x - ct$, $x_0 = x + ct$ the x-axis, and hence the area of this triangle is $\frac{1}{2c}$. Now, substituting (d) in (c), (and imposing our boundary conditions to find that there are no extra terms) yields:

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-ct}^{x+ct} f(x',t')dx'dt' = \int \int_{\Delta} f(y,s)dyds$$

Question 6

Exercise. 1.6:

Consider the PDE with boundary conditions:

$$\begin{aligned} u_{tt} - c^2 u_{xx} + \omega^2 u &= 0, \qquad 0 < x < L, \\ (u_x - \alpha u_t)(0,t) &= 0, \\ (u_x + \beta u_t)(L,t) &= 0, \end{aligned}$$

where $\alpha > 0, \beta > 0$ are constants. Prove that the energy $E(t)$ defined as:

$$E(t) = \frac{1}{2} \int_0^L (u_t^2 + c^2 u_x^2 + \omega^2 u^2)dx$$

is a non-increasing function of t.

Proof. We first compute $dE(t)/dt$:

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{1}{2} \frac{d}{dt} \int_0^L \left(u_t^2 + c^2 u_x^2 + \omega^2 u^2\right) dx \\ &= \frac{1}{2} \int_0^L \frac{\partial}{\partial t} \left(u_t^2 + c^2 u_x^2 + \omega^2 u^2\right) dx \\ &= \int_0^L \left(u_t u_{tt} + c^2 u_x u_{xt} + \omega^2 u u_t\right) dx \end{aligned}$$

From here, let us integrate by parts the $c^2 u_x u_{xt} = c^2 u_x u_{tx}$ term:

$$\int_0^L c^2 u_x u_{xt} = c^2 \left(u_x u_t\big|_0^L\right) - c^2 \int_0^L u_{xx} u_t dx$$

Thus, plugging this back into our most recent equation yields:

$$\begin{aligned} \frac{dE(t)}{dt} &= c^2 \left(u_x u_t\big|_0^L\right) + \int_0^L \left(u_t u_{tt} - c^2 u_t u_{xx} + \omega^2 u u_t\right) dx \\ &= c^2 \left(u_x u_t\big|_0^L\right) + \int_0^L \left(u_t(u_{tt} - c^2 u_{xx}) + \omega^2 u u_t\right) dx \\ &= c^2 \left(u_x u_t\big|_0^L\right) + \int_0^L \left(u_t(-\omega^2 u) + \omega^2 u u_t\right) dx \qquad \text{since } u_{tt} - c^2 u_{xx} = -\omega^2 u \\ &= c^2 \left(u_x u_t\big|_0^L\right) + 0 \\ &= c^2 \left(u_x(L,t)u_t(L,t) - u_x(0,t)u_t(0,t)\right) \end{aligned}$$

And thus by recalling our boundary conditions of $(u_x - \alpha u_t)(0,t) = 0$ and $(u_x + \beta u_t)(L,t) = 0$:

$$\begin{aligned}
&\frac{dE(t)}{dt} = c^2\Big(- \beta\big(u_t(L,t)\big)^2 - \alpha\big(u_t(0,t)\big)^2\Big) \\
\Rightarrow &\frac{dE(t)}{dt} \leq 0 \quad \text{since } c^2, (u_t(L \text{ or } 0,t))^2 \geq 0, \text{ and } -\alpha, -\beta < 0
\end{aligned}$$

I.e., $E(t)$ is a non-increasing function of t. □

Question 7

Exercise. 1.7:

Let $f(x)$ be any C^2-function defined on $\mathbb{R}^3$ that vanishes outside a disk centered at the origin. Prove that

$$4\pi f(0) = \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}|}\Delta f(\mathbf{x}) d\mathbf{x}$$

Proof. From the question, we know that $f \equiv 0\ \forall x \notin B(0,R)$ for some $R \in \mathbb{R}^+$. Hence, let us set up the domain $U_\epsilon := B(0,2R)\backslash B(0,\epsilon)$. We now invoke Green's Theorem (we can do so since both f and $1/|\mathbf{x}|$ are C^2 on $U_\epsilon\ \forall \epsilon > 0$) as follows:

$$\begin{aligned}
\int_{U_\epsilon} \frac{1}{|\mathbf{x}|}\Delta f(\mathbf{x}) d\mathbf{x} &= \int_{U_\epsilon} \left(\frac{1}{|\mathbf{x}|}\Delta f(\mathbf{x}) - \Delta\left(\frac{1}{|\mathbf{x}|}\right) f(\mathbf{x}) \right) d\mathbf{x} && \text{since } \Delta(1/|\mathbf{x}|) = 0\ \forall x \in U_\epsilon \\
&= \int_{\partial U_\epsilon} \left(\frac{1}{|\mathbf{x}|}\frac{\partial f(\mathbf{x})}{\partial n} - f(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}|}\right)\right) dS && \text{by the Divergence Theorem} \\
&= \int_{|\mathbf{x}|=\epsilon} \left(\frac{1}{|\mathbf{x}|}\frac{\partial f(\mathbf{x})}{\partial n} - f(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}|}\right)\right) dS && \text{since } f \text{ and } \partial_n f \equiv 0 \text{ for } |\mathbf{x}| \geq 2R \\
&= \int_{|x|\geq\epsilon} \frac{1}{|\mathbf{x}|}\Delta f(\mathbf{x}) d\mathbf{x}
\end{aligned}$$

From here, we naturally let $\epsilon \to 0$. Doing so gives us the following:

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}|}\Delta f(\mathbf{x}) d\mathbf{x} &= \lim_{\epsilon\to 0}\int_{|\mathbf{x}|=\epsilon} \left(-\frac{1}{|\mathbf{x}|}\frac{\partial f(\mathbf{x})}{\partial |\mathbf{x}|} - f(\mathbf{x})\frac{\partial}{\partial |\mathbf{x}|}\left(\frac{1}{|\mathbf{x}|}\right)\right) dS && \text{since } \partial_n = -\partial_{|\mathbf{x}|} \text{ on } |\mathbf{x}| = \epsilon \\
&= -\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\int_{|\mathbf{x}|=\epsilon} \frac{\partial f(\mathbf{x})}{\partial |\mathbf{x}|} dS - \frac{1}{\epsilon^2}\int_{|\mathbf{x}|=\epsilon} f(\mathbf{x}) dS \right) \\
&= -\lim_{\epsilon\to 0}\left(\frac{4\pi\epsilon^2}{\epsilon}\text{Avg}_{|\mathbf{x}|=\epsilon}\left(\frac{\partial f(\mathbf{x})}{\partial |\mathbf{x}|}\right) - \frac{4\pi\epsilon^2}{\epsilon^2}\text{Avg}_{|\mathbf{x}|=\epsilon}\left(f(\mathbf{x})\right)\right) \\
&= 4\pi f(0)
\end{aligned}$$

□

Question 8

Exercise. 1.8:

(a) Write the Green's function of the Laplace operator with Dirichlet boundary condition in the half space

$$D = \{(x, y, z) \in \mathbb{R}^3, z > 0\}$$

Explain your construction.

(b) Use the Green's function to write the solution of the Laplace equation $\Delta u = 0$ in D with the boundary condition $u(x, y, 0) = h(x, y)$.

(a) If we define $\tilde{\mathbf{x}} := (x_1, x_2, -x_3)$, then we claim that for this example, Green's function will be:

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{-1}{\omega_3}\left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} - \frac{1}{|\mathbf{x} - \tilde{\mathbf{x}}_0|}\right)$$

We choose this construction since the fundamental solution of u is $u(\mathbf{x}, \mathbf{x}_0) = \frac{-1}{\omega_3|\mathbf{x}-\mathbf{x}_0|}$, and $\lim_{|x|\to\infty} G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}, \mathbf{x}_0)|_{z=0} = 0$.

(b) From Theorem 12 in the textbook, the solution will take on the form:

$$u(x) = -\int_{\partial D} h(x, y)\frac{\partial G(x, y)}{\partial n} dS$$

And since $\partial_n G = -G_{x_3} = \frac{-2x_3}{|\mathbf{x}-\mathbf{x}_0|^3}$, we can see that:

$$u(x) = -\frac{2x_3}{\omega_3}\int_{\partial D} \frac{h(x, y)}{|\mathbf{x} - \mathbf{x}_0|^3} dS$$

Question 9

Exercise. 1.9:

The bilaplacian operator Δ^2 is defined by $\Delta^2 u = \Delta(\Delta u)$. Consider the equation:

$$\Delta^2 u = 0$$

in $\mathbb{R}^2$

(a) Prove that:

$$v(r) = \frac{1}{8\pi} r^2 \ln(r)$$

where $r = |\mathbf{x}|$ is the radial coordinate, satisfies $\Delta^2 v = 0$ for all $\mathbf{x} \neq 0$.

(Hint: Note that $\Delta v = (1 + \ln r)/(2\pi))$

(b) Let D be a bounded, open, connected domain of $\mathbb{R}^2$. Show that for all $u \in C^4(D) \cap C(\overline{D})$ satisfying $\Delta^2 u = 0$, and $\mathbf{x_0} \in D$, one has the representation formula:

$$u(\mathbf{x_0}) = -\int_{\partial D}\left(v\frac{\partial}{\partial n}\Delta u - \Delta u\frac{\partial v}{\partial n} + \Delta v\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\Delta v\right) dS$$

(a) Since I didn't read the question correctly before typing up my solution (argh), we see if (a) holds for $n \geq 2$ dimensions, and see that it actually holds $\iff n = 2$...

Our plan will be to convert the Laplacian, which is a function of $x_1, \dots, x_n$'s 2nd derivatives, simply into a function of r. If $\mathbf{x} = (x_1, \dots, x_n)$, then in n-dimensional polar coordinates, we will need the variables: r and $\theta_1, \dots, \theta_{n-1}$ to make the "conversion" into polar. This would be quite the daunting task; however, in this exercise, we can make the observation that since v is a function of only r, $\partial_{\theta_i} v = 0 \ \forall i \in \{1, \dots, n-1\}$. Therefore, Δv will be very easy to calculate in finite dimensions. Explicitly, the n-dimensional Δ takes the form:

$$\Delta(u) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial u}{\partial r} \right)$$

And hence:

$$\begin{aligned}
\Delta(\Delta v) &= \frac{\partial^2 u}{\partial x_i^2} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial (1 + \ln(r))/(2\pi)}{\partial r} \right) \\
&= \frac{n-2}{2\pi r^2} \\
&= 0 \iff n = 2
\end{aligned}$$

(b) *Proof.* To prove this identitiy, we make use of Green's Second Identity, which states for $\psi, \varphi \in C^2(\Omega)$:

$$\int_{\Omega} (\psi \Delta \varphi - \varphi \Delta \psi) d\mathbf{x} = \int_{\partial \Omega} \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) dS$$

Thus, let us apply Green's Second Idenity twice, both where $\Omega = \hat{D} \subset D$, where one has $(\varphi, \psi) = (\Delta u, v)$ and the other where $(\varphi, \psi) = (\Delta v, u)$:

$$(\varphi, \psi) = (\Delta u, v) \quad \int_{\hat{D}} (\Delta u \Delta v - v \Delta(\Delta u)) d\mathbf{x} = \int_{\partial \hat{D}} \left(\Delta u \frac{\partial v}{\partial n} - v \frac{\partial \Delta u}{\partial n} \right) dS \tag{4}$$

$$(\varphi, \psi) = (\Delta v, u) \quad \int_{\hat{D}} (\Delta v \Delta u - u \Delta(\Delta v)) d\mathbf{x} = \int_{\partial \hat{D}} \left(\Delta v \frac{\partial u}{\partial n} - u \frac{\partial \Delta v}{\partial n} \right) dS \tag{5}$$

Now, let us subtract (5) from (4), which implies (after cancelling the $\Delta u \Delta v$ terms) that:

$$\int_{\hat{D}} (v \Delta^2 u) d\mathbf{x} + \int_{\hat{D}} (u \Delta^2 v) d\mathbf{x} = - \int_{\partial \hat{D}} \left(\Delta u \frac{\partial v}{\partial n} - v \frac{\partial \Delta u}{\partial n} - \Delta v \frac{\partial u}{\partial n} + u \frac{\partial \Delta v}{\partial n} \right) dS$$

However, since $\Delta^2 u$ from the question, we actually have:

$$\int_{\hat{D}} (u \Delta^2 v) d\mathbf{x} = - \int_{\partial \hat{D}} \left(\Delta u \frac{\partial v}{\partial n} - v \frac{\partial \Delta u}{\partial n} - \Delta v \frac{\partial u}{\partial n} + u \frac{\partial \Delta v}{\partial n} \right) dS$$

We now note that as long as $\hat{D} \subset D$, the above is true. Also, our only assumption about v thus far has been $v \in C^2(D)$. Thus, let us choose $\hat{D} = D \backslash D_r(\mathbf{x_0})$, and make v depend on $\mathbf{x_0}$ and r in that $\Delta^2 v(\mathbf{x}) = 0 \ \forall \mathbf{x} \notin B_r(\mathbf{x_0})$, and $\Delta^2 v(\mathbf{x_0}) = 1$. From this construction, $\lim_{r \to 0} \hat{D} = D$, and $\lim_{r \to 0} \int_{\hat{D}} (u \Delta^2 v) d\mathbf{x} = u(\mathbf{x_0})$, and hence we have now shown:

$$u(\mathbf{x_0}) = - \int_{\partial D} \left(v \frac{\partial}{\partial n} \Delta u - \Delta u \frac{\partial v}{\partial n} + \Delta v \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \Delta v \right) dS$$

□

Question 10

Exercise. 1.10:

The Fourier Sine series of $f(x) = x$ on the interval $(0, l)$ is:

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

where $A_n = (-1)^{n+1}\left(\frac{2l}{n\pi}\right)$.

(a) Write the Parseval's equality.

(b) Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(a) We assume that for this question, we are to write the most applicable version of Parseval's equality for computing part (b). Thus, if $f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$ for some A_n, then:

$$\frac{2}{l}\int_0^l (f(x))^2 dx = \sum_{n=1}^{\infty} A_n^2$$

(b) If we let $l = \pi$, then:

$$A_n = (-1)^{n+1}\left(\frac{2l}{n\pi}\right) = (-1)^{n+1}\left(\frac{2}{n}\right)$$

And thus by Parseval's Equality:

$$\begin{aligned}
\sum_{n=1}^{\infty}\left((-1)^{n+1}\left(\frac{2}{n}\right)\right)^2 &= \frac{2}{\pi}\int_0^{\pi}(x)^2 dx \\
\iff \sum_{n=1}^{\infty}\left((1)\left(\frac{4}{n^2}\right)\right) &= \frac{2}{\pi}\left(\left(\frac{x^3}{3}\right)\Big|_0^{\pi}\right) \\
\iff \sum_{n=1}^{\infty}\frac{1}{n^2} &= \left(\frac{2}{4\pi}\right)\left(\frac{\pi^3}{3}\right) \\
\iff \sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{6}
\end{aligned}$$