MAT357 Problem Set

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Part I

PS 0

1 Problem Set 0

Questions to do:
7, 8, 12, 14, 15, 18, 23, 24, 27, 29, 30, 32, 34, 39, 42, 43, 48, 96, 97, 107, 127, 129, 130, 131

1.1 Chapter 2, Problem 8

1.2 Chapter 2, Problem 12

1.3 Chapter 2, Problem 14

1.4 Chapter 2, Problem 15

1.5 Chapter 2, Problem 18

1.6 Chapter 2, Problem 23

1.7 Chapter 2, Problem 24

1.8 Chapter 2, Problem 27

1.9 Chapter 2, Problem 29

1.10 Chapter 2, Problem 30

1.11 Chapter 2, Problem 32

1.12 Chapter 2, Problem 34

1.13 Chapter 2, Problem 39

1.14 Chapter 2, Problem 42

1.15 Chapter 2, Problem 43

1.16 Chapter 2, Problem 48

1.17 Chapter 2, Problem 96

1.18 Chapter 2, Problem 107

1.19 Chapter 2, Problem 127

1.20 Chapter 2, Problem 129

1.21 Chapter 2, Problem 130

1.22 Chapter 2, Problem 131

Part II
PS1

2 Homework

2.1 Chapter 2, Problem 54

We Prove this for $\forall(M, d')$ metric space, and this will be true since $\mathbb{R}$ with some metric is metric space. We will also use the metric space (S,d) given in the question for the domain.

A since f is uniformly continuous then we know

$$\forall \epsilon \exists \delta > 0 \text{ st. } \forall x, y \in M \; d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon$$

- **Defining $\bar{f}$**: $\bar{f}$ as function extending f, such that when $s \in S$ then $\bar{f}(s) = f(s)$. And when $s = \lim(s_n)$ st. $s \in \partial S$ then $\bar{f}(s) = \lim(f(s_n))$
- **Proving $\bar{f}$ is well defined** We can note that by the fact that limits are unique , $\bar{f}(s)$ is unique for $s \in \partial S$ and so it is well defined given this condition and the conditions of its construction
- **Proving $\bar{f}$ is uniformly continuous**
  Since f is uniformly continuous, then we know for any two points in S, $\bar{f}$ is also uniformly continuous
  Thus, we pick $\hat{x}, \hat{y} \in \bar{S}$ where both $\hat{x}$ and $\hat{y}$ are limit points of $(x_n), (y_m) \in S$
  Want to show that

  $$\forall \epsilon \exists \delta > 0 \text{ st. } \forall x, y \in \bar{S} \; d(\hat{x}, \hat{y}) < \delta \Rightarrow d'(\bar{f}(\hat{x}), \bar{f}(\hat{y})) < \epsilon$$

  Then we can use the fact that f is uniformly continuous and that $\hat{x}, \hat{y}$ are limit points to note that:

  $$\exists \delta' \text{ st. } d'(\bar{f}(x_n), \bar{f}(\hat{x})) + d'(\bar{f}(y_n), \bar{f}(\hat{y})) \leq \frac{\epsilon}{2} \text{ when } d(x_n, x) + d(y, y_n) \leq \frac{\delta'}{2}$$

  (since $\hat{x}, \hat{y}$ are limit points, we can get the sequences that converge to them as close as we would like by setting find for enough in the the given sequence)

  We can also note that by uniform continuity of f, and the fact that we have extended it to $\bar{f}$:

  $$\exists \delta'' \text{ st. } d'(\bar{f}(x_n), \bar{f}(y_n)) = d'(f(x_n), f(y_n)) < \frac{\epsilon}{2} \text{ when } d(x_n, y_n) \leq \frac{\delta''}{2}$$

  And since:

  $$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) \leq d(x, y) + \frac{\delta}{2}'$$

  By setting $d(x,y) \leq |\frac{\delta''}{2} - \frac{\delta'}{2}|$ we get the result we looked for above

  Thus we set $\delta = \min\{\frac{\delta'}{2}, \frac{\delta''}{2}\}$ and thus we note that when

  $$d(\hat{x}, \hat{y}) \leq d(x_n, \hat{x}) + d(x_n, y_n) + d(y_n, \hat{y}) \leq \delta \leq \frac{\delta'}{2} + \frac{\delta''}{2}$$

Then

$$d'(\bar{f}(\hat{y}), \bar{f}(\hat{x})) \leq d'(\bar{f}(\hat{x}), \bar{f}(x_n)) + d'(\bar{f}(y_n), \bar{}(x_n)) + d'(\bar{f}(y_n), \bar{f}(\hat{y})) \leq \epsilon$$

B Given $\bar{f} : \bar{S} \to M$, assume $\exists g : \bar{S} \to M$
Both of which are continuous extensions of f
We then pick $s \in \bar{S}$ st. $s = \lim(s_n)$ where (s_n) is a sequence in S.
By Continuity we can have that: $\star$

$$\bar{f}(s_n) \to \bar{f}(s) \text{ as } s_n \to s$$

$$g(s_n) \to g(s) \text{ as } s_n \to s$$

Then notice that:

$$d'(\bar{f}(s), g(s)) \leq d'(\bar{f}(s), \bar{f}(s_n)) + d'(\bar{f}(s_n), g(s_n)) + d'(g(s_n), g(s))$$

Now note that:

(a) $\bar{f}(s_n), g(s_n) \in S$ which implies, since they are both extensions of f, $\bar{f}(s_n) = f(s_n) = g(s_n)$

$$\Rightarrow d'(\bar{f}(s), g(s_n)) = d'(f(s), f(s_n)) = 0$$

(b) by $\star$ and the fact these are both uniformly continuous function, we know that we can set $d'(\bar{f}(s), \bar{f}(s_n)) + d'(g(s), g(s_n)) \leq \epsilon$

As a result given these two points, as two sub sequences converge in $\bar{S}$ the distance between $\bar{f}(s)$ and $g(s)$ converges to 0, and so for each such given point, the two function are the same.

2.2 Chapter 2, Problem 108

I would like to prove this in 3 parts. (A) id is continuous (B) id is Bijective (C) inverse of id is not continuous

A Pick $\epsilon > 0$ then set $\delta = \epsilon$ Then by setting $max_{[0,1]}|f - g| < \delta$ We get that

$$\int_0^1 |f - g|dx \leq max_{[0,1]}|f - g| \leq \delta = \epsilon$$

B $\forall f, g \in C([0, 1], \mathbb{R})$ if $id(f) = id(g)$ then since

$$f = id(f) = id(g) = g \text{ then } f = g$$

This gives us that the id mapping is injective. Thus, since the set of the metric space of both the range and domain are the same, and the mapping is injective, we know that it is also surjective. Thus, the mapping is bijective.

C Since we have that $\forall f, g \in C([0, 1], \mathbb{R})$, $max_{[0,1]}|f - g| \geq \int_0^1 |f - g|dx$
Then $\nexists \delta \forall \epsilon$ st. $\int_0^1 |f - g|dx \leq \delta \Rightarrow max_{[0,1]}|f - g| \leq \epsilon$
For example:
set $(g_n) = (x - 1)^n$ and set $f(x) = 0$
Then we can notice that even though $(g_n) \to f$ in $C_{int}([0, 1].\mathbb{R})$
The image of the sequence under the inverse of id, $id^{-1}(g_n) = (g_n)$, does not converge to f since:

$$max_{[0,1]}|f - g_m| = 1 \text{ for all } g_m \in (g_n)$$

And so the inverse of the id function is not continuous.

Part III
Problem Set 2

3 Homework

3.1 Chapter 2, Problem 44

Question:Consider function $f : M \to \mathbb{R}$. Its graph is the set

$$\Gamma_f = \{(p, y) \in M \times \mathbb{R} : y = fp\}$$

A **Prove that if f is continuous then its graph is closed (as subset of $M \times \mathbb{R}$)**

Proof. To show that Γ_f is closed, we will show that Γ_f^c is open.
Pick any $t \in M \times \mathbb{R}$ such that $t \in \Gamma_f^c$ then let $t = (p', y')$ where $p' \in M, y' \in \mathbb{R}$. Then $\exists t' = (p', f(p')) = (p', y)$ and by continuity of f :

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall p'' \in M, d(p'', p') < \delta \Rightarrow d(f(p''), f(p')) < \epsilon$$

Then if we set $\omega = d(y, y')$ then we can solve for a δ such that $|f(p''), y| < \frac{\omega}{3}$. and by setting $\lambda = \min\{\delta, \frac{\omega}{3}\}$, then we have a ball in $B_\lambda(t) \in \Gamma_f^c$ □

B **Prove that if f is continuous and M is compact then its graph is compact**

Proof. Set $\Gamma_f = \{(p, y) \in M \times \mathbb{R} | y = f(p)\}$ and pick a sequence (α_n) in Γ_f.

Then we know that $\forall a_n \in (\alpha_n)$ $a_n = (p_n, y_n)$ such that $y_n = f(p_n)$. Then, since M is compact, $\exists$ a subsequence $(p_{n_k}) \to p \in M$ that corresponds with a sequence in Γ_f, specifically (a_{n_k}). And since the subsequence in the domain converges and the function is continuous, it also converges in the image of the function, i.e. $f(p_{n_k}) \to f(p)$.

Thus, we could set $(p, f(p)) = a$ and notice that since $p = f(p)$ and both points are in the domain of Γ_f, then the subsequence $(a_{n_k}) \to a \in \Gamma_f$. □

C **Prove that if the graph of f is compact then f is continuous**

Proof. Pick a convergent sequence (a_n) in M, where $(a_n) \to a \in M$, and let (b_n) be the corresponding sequence in the image, where $b_n = f(a_n)$. and let b be the point that image converges to when (a_n) converges (we assume M is closed and so the $a \in M$).
Notice: The sequence of the image either

- Converges to $f(a)$ (i.e. $f(b) = f(a)$) and so we have a trivial solution
- Diverges: This can not happen since:
  If $f(a_n)$ diverges as $a_n \to a$ then we have that

$$\exists \epsilon^\star, \forall N, st|f(a_n) - f(b)| > \epsilon \text{ when } n > N, \forall f(b)$$

  However,this implies that if we choose a covering of balls whose radius is less then ϵ, the covering can not be finite, and thus the graph is not compact.

- Converges to $f(b)$, where $f(b) \neq f(a)$ (proof of the contradiction to this can be found bellow)

Then there is a corresponding sequence in Γ_f , specifically (γ_n) where $\gamma_n = (a_n, b_n)$. Since (γ_n) is a sequence in the graph, then the sequence has convergent subsequence $(\gamma_{n_k}) \to \gamma$ which corresponds to subsequences (a_{n_k}) and their image (b_{n_k}) where $(a_{n_k}) \to a'$ and $(b_{n_k}) \to b' = f(a')$ where $\gamma = (a', b')$. However, since (a_{n_k}) is a subsequence of (a_n) and both converge, it must be that $a = a'$, and since we know that (b_{n_k}) is subsequence of (b_n) and both converge, it implies that $b = b' = f(a)$. And so, we have now that, given a convergent sequence (a_n) in M, the image of the convergent sequence converges to the image of the point convergence of the sequence in the domain, which means that f preserves sequential convergence, and hence, is continuous.

□

D **What if the graph is merely closed? Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed**
Consider this function for a counter example:

$$f(x) = \begin{cases} \frac{1}{p} & x \in \mathbb{Q}, x = \frac{p}{q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

3.2 Chapter 2, Problem 76 a-c

A **The intersection of connected sets need not be connected. Give an example.**
Solution: Consider the function

$$\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\} \cap \{(x, y) \in \mathbb{R}^2 | (x - 1)^2 + y^2 = 1\}$$

Both sets are closed unit circles, centered at different points in $\mathbb{R}^2$, but their intersection is two disconnected points in the plane.

B **Supposed that $S_1, S_2, S_3, ...$ is a sequence of connected, closed subsets of the plane and $S_1 \supset S_2 \supset ....$ Is $S = \cap S_n$ connected ? Give a proof or counterexample**
Solution: Consider this function as a counter example:

$$S_n = \mathbb{R}^2 \backslash \{(x, y) \in \mathbb{R}^2 | -1 < y < 1, x \in (-n, n)\}$$

which is the plane dissected by and open rectangle, meaning that all sets are closed, and yet their intersection is are the two parts of the plane that are "cut" by the rectangle.

C **Does the answer change if the sets are compact**

Proof. We prove this through these steps

(a) Set $S = \cap_{i=1}^{\infty} S_i$ (where S_i are each compact and connected

(b) S is compact, and by assumption is disconnected. Which implies that $\exists P, P^c$ clopen such that$P^c U P = S$, and by inheritance, are both compact and have disjoint covers G, F respectively.

(c) We then set $T_i = S_i \backslash F \cup G$ and it follows that $\cap_{i=1}^{\infty} T_i = \emptyset$
(since the intersection of T_i's is equal to $S \backslash F \cup G = \emptyset$)
(also note that this holds because T_i is compact for all i's)

(d) This implies that $\exists S_i$ such that $S_i \subset F \cup G$, and since $P \cap S_i \neq \emptyset$ and $P^c \cap S_i \neq \emptyset$, but this is a contradiction, since S_i was assume to be connected.

□

3.3 Chapter 2, Problem 118 a-b

The implications of compactness are frequently equivalent to it. Prove:

A **If every continuous function $f : M \to \mathbb{R}$ is bounded then M is compact.**

Proof. (using contra-positive) Assume M is not a compact Metric Space.

Then pick any sequence $(x_n) \subseteq M$ then all its subsequence do not converge, and so we can assume that each element $x_i \in (x_n)$ is distinct, and has ball of radius ϵ_i around it, such that $x_j \notin M_{\epsilon_i}$ iff $x_j \neq x_i$. We can also note that given a sequence of points within each ball around each element in (x_n) has no convergent subsequence (i.e. (q_n) is the set of points $q_i \in M_{\epsilon_i}(x_i) \forall i$, does not have a converging subsequence).Then we define the function $f(x)$ as

$$f(x) = \begin{cases} \frac{\epsilon_n - d(x,x_n)}{a_n + d(x_n,x)} & x \in M_{\epsilon_n}(x_n) \\ 0 & \text{Otherwise} \end{cases}$$

Where $a_n > 0$. Thus we notice the following:

(a) f **is continuous:**
Given $\lambda > 0$, if we pick $\delta \geq \lambda \left| \frac{a_n^2}{(\epsilon_n - a_n)} \right|$ then $d(q', q) < \delta$ implies that $|f(q') - (q)| =$

$$\begin{aligned} \left| \frac{\epsilon_n - d(q,x_n)}{a_n + d(x_n,q)} - \frac{\epsilon_n - d(q',x_n)}{a_n + d(x_n,q')} \right| &= \left| \frac{(\epsilon_n - a_n)(d(q,x_n) - d(q',x_n))}{(a_n + d(x_n,q))(a_n + d(x_n,q'))} \right| \\ &\leq \left| \frac{(\epsilon_n - a_n)\delta}{(a_n + d(x_n,q))(a_n + d(x_n,q'))} \right| \\ &\leq \left| \frac{(\epsilon_n - a_n)\delta}{a_n^2 + a_n(d(x_n,q) + d(x_n,q')) + d(x_n,q)d(x_n,q')} \right| \\ &\leq \left| \frac{(\epsilon_n - a_n)\delta}{a_n^2} \right| \\ &\leq \lambda \end{aligned}$$

(b) **f is unbounded:** Given bound M, if we set $a_n > \frac{\epsilon_n}{M}$ then we have that when $x = x_n$ (for any n) then

$$\begin{aligned} f(x) &= \frac{\epsilon_n - d(x,x_n)}{a_n + d(x_n,x)} \\ &= \frac{\epsilon_n}{a_n} \\ &\geq M \end{aligned}$$

Thus, There exists an unbounded continuous function.

□

B **If every continuous bounded function $f : M \to \mathbb{R}$ achieves a maximum or minimum then M is compact.**

Proof. Continuing with the same function from the question above:

- f **is continuous and bounded**
  Since the image of f is in $\mathbb{R}$ and it is bounded, then we know that $\exists$ a superemum, and we define it to be M. then if we set $a_n < \frac{\epsilon_n}{M}$ then we have that at its maximum point, when $x = x_n$ (for any n),

$$\begin{aligned} f(x) &= \frac{\epsilon_n - d(x, x_n)}{a_n + d(x_n, x)} \\ &= \frac{\epsilon_n}{a_n} \\ &< M \end{aligned}$$

  Thus there exists a bounded continuous function which does not achieve its maximum.

□

3.4 Chapter 2, Problem 140

Consider the Hilbert cube

$$H = \{(x_1, x_2, ...) \in [0,1]^\infty : \forall n \in \mathbb{N} \textbf{ we have } |x_n| \leq \frac{1}{2^n}\}$$

Prove that H is compact with respect to the metric

$$d(x,y)) = \sup_n |x_n - y_n|$$

where $x = (x_n), y = (y_n)$. [Hint: Sequences of sequences]
Remark: Although compact, H is infinite- dimensional and is homeomorphic to no subset of $\mathbb{R}^m$

Proof. Given a a sequence (x_n) in H, we observe that:

A Every dimension of each element of the sequence is bounded

B The bounds of each dimension are getting smaller

C H is closed

To simplify the notation of the proof, we will take a moment to define the following:

- a subsequence of a sequence (x_{n_i}) is $(x_{n_{i+1}})$ (and if $i = 0$ then we will omit it)
- the point of convergence of a sequence (x_{n_i}) is j_i and to show convergence we will state $(x_{n_i}) \to j_i$
- the k^{th} element and m^{th} component of that element, in the sequence (x_{n_i}) will be denoted $x^{k,m}_{n_i}$ (m will be omitted when just considering the elements of sequence without their elements)

Now we use the first observation (A), to note that the first component of each element in (x_n) is bounded, and thus, $\exists(x_{n_1})$, subsequence of (x_n), for which $x^{k,1}_{n_1} \to j_1$. Then, we again find that $\exists(x_{n_2})$ for which $x^{k,2}_{n_2} \to j_2$. We can then continue taking subsequence of each subsequence to get (x_{n_t}), subsequence of $(x_{n_{t-1}})$, for which $x^k_{n_t}, t \to j_t$.
We can notice that since H is closed, all the points of convergence are within H, and we set:

- $(g_n) = \{x^l_{n_l}\}^\infty_{l=1}$(diagonalizing the subsequences)
- $j = (j_1, j_2, j_3, \ldots)$

Thus we notice that (g_n) is a subsequence of (a_n) and is also of all the subsequence constructed above, and thus by picking N large enough $d(g_s, p) < \epsilon \; \forall k > N$

This is given since:

A by picking N large enough, we can get the components of g_s, which are made of elements in subsequences which converge to the elements of j, as close as we would like them to be (thus $\sup_{k>N} |g^n_k - j_n| \to 0$ as $k \to \infty$)

B by picking N large, we get the all the components above order N, of the elements g_s, where $s > N$, to be bounded by $\frac{1}{2^N}$ and so $\sup_{s>N} |g^n_k - j_k| < \frac{1}{2^N}$,

And thus, considering these two ideas, we can choose N accordingly for any given ϵ. And thus (g_n) is a converging subsequence which, by its closure, converges within H. □

3.5 Chapter 2, Problem 147 a-c

Let M be metric space let $\mathcal{K}$ denote the class of nonempty compact subsets of M. The r - neighborhood of $A \in \mathcal{K}$ is

$$M_r A = \{x \in M : \exists a \in A \text{ and } d(x,a) < r\} = \cup_{a \in A} M_r a$$

For $A, B \in \mathcal{K}$ define

$$D(A,B) = inf\{r > 0 : A \subset M_r B \text{ and } B \subset M_r A\}$$

A **Show that D is a metric on $\mathcal{K}$ (It is called the Hausdorff metric and $\mathcal{K}$ is called the hyperspace of M)**

Proof. To prove that D is a metric we will prove that

- (1) $d(X,Y) \geq 0$ and $D(x,y) = 0$ iff $x = y$
- (2) $d(X,Y) = d(Y,X)$
- (3) $d(X,Y) \leq d(X,Z) + d(Z,Y)$

(a) $d(X,Y) \geq 0$ by construction since $r > 0$. Thus, we are left to show that $d(X,Y) = 0 \Leftrightarrow X = Y$.
Forward direction, if $d(X,Y) = 0$, then we know have that $X \in M_0(Y)$ and $Y \in M_0(X)$ $\Rightarrow X = Y$.
Backward direction if $X = Y$ then for any $\epsilon > 0$ $M_\epsilon(X) \supset Y$ and $M_\epsilon(Y) \supset X$ $\Rightarrow d(X,Y) = 0$

(b) $r = d(X,Y) \Rightarrow X \subseteq M_r(Y)$ and $Y \subseteq M_r(X) \Leftrightarrow Y \subseteq M_r(X)$ and $\Rightarrow X \subseteq M_r(Y) \Rightarrow d(Y,X) = r$

(c) Let $d(X,Y) = \lambda, d(X,Z) = \phi, d(Z,Y) = \omega$ $(\star)$
Pick any $x \in X$ then, and notice that, by $\star$, $\exists y \in Y$ such that $d(x,y) < \lambda$, and $\exists z \in Z$ such that $d(z,y) < \omega$
No since we know that $d(\cdot,\cdot)$ is a metric, then we have $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x \in X, y \in Y, z \in Z$, which implies that

$$D(X,Y) \leq D(X,Z) + D(Z,Y)$$

□

B **Denote by $\mathcal{F}$ the collection of finite nonempty subsets of M and prove that $\mathcal{F}$ is dense in $\mathcal{K}$. That is, given $A \in \mathcal{K}$ and given $\epsilon > 0$ show $\exists F \in \mathcal{F}$ such that $D(A,F) < \epsilon$**

Proof. Given $A \in \mathcal{K}$ and $\epsilon > 0$.
Then, since A is compact then it is totally bounded. this implies that A is covered by $C = \cup_{i=1}^n M_\epsilon(x_i)$ where $x_i \in A$. Then setting $X = \{x_i\}_{i=1}^n$, which is a finite set of elements in M (and thus non empty), then $X \subseteq \mathcal{F}$.

Thus, we have that X is in $\mathcal{F}$ and since it is points in the covering of A then $x \subset M_\epsilon A$, and by construction, we have that $A \subset M_\epsilon X$. Therefore, $D(A,X) < \epsilon$ □

C **If M is compact prove that $\mathcal{K}$ is compact**

4 Bonus

Proof that:
There is no metric on $M = \{\text{continues functions } f[a,b] \to \mathbb{R}\}$ with property that

$$f_n(x) \to f(x) \forall x \in [a,b] \Leftrightarrow d_e(f_n, f) \to 0$$

i.e. continuous functions with topological point wise convergence is not a metric space