MAT357 Notes

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Winter 2017

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Part I
Topology

1 Metric Space

A metric space is a Set M with elements, referred to as points of M, together with a metric, d(x,y) , that has the following 3 properties:

1. $d(x, y) = d(y, x)$
2. $d(x, y) > 0 \Leftrightarrow x \neq y$
3. $d(x, y) \leq d(x, z) + d(z, y)$

One can think of $d(x, y)$ as "distance from x to y"
Examples:

1. Set $M = R^n$
   Possible distances:
   $$(\mathbf{A})\ d_e(x, y) = \sqrt{\sum_{i=0}^{n}(x_i - y_i)^2}$$
   $$(\mathbf{B})\ d_{max}(x, y) = max|x_i - y_i|$$
   $$(\mathbf{C})\ d_{sum}(x, y = \sum |x_i - y_i|$$
2. Set $M = \{\text{continuous function}[a, b] \to \mathbb{R}\}$
   Possibel metrics:
   $$d_e(f, g) = \sqrt{\int_a^b (f(x) - g(x))^2 dx}$$
   $$d_{max}(f, g) = \begin{matrix} sup \\ x \in [a, b] \end{matrix}\ |f(x) - g(x)|$$
   $$d_{sum}(f, g) = \int_a^b |f(x) - g(x)| dx$$
3. M is any set
   $$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & o.w. \end{cases}$$
   this is called the "discrete metric"
4. subsets of a metric space e.g. subsets of (R^n, d_e)

Proof that the first metric and the set example 2 create a metric space

1. trivial since by def this holds
2. obvious since the integral will always be positive

3. Proof bellow

   - Note the Cauchy Schwartz inequality

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f(x)^2dx \int_a^b g(x)^2dx$$

   Set $f, g, h \in M$

$$d_e(f,g)^2 = \int_a^b (f-g)^2dx$$

$$= \int_a^b ((f-h)+(h-g))^2dx = \int_a^b (f-h)^2dx + \int_a^b (h-g)^2dx + 2\int_a^b (f-h)(h-g)dx$$

$$\leq d_e(f,h) + d_e(h,g) + \int_a^b (h-g)^2dx + 2\sqrt{\int_a^b f(x)^2dx \int_a^b g(x)^2dx}$$

$$= (d_e(f,h) + d_e(h,g))^2$$

1a) Balls

A Ball is denotes by $M_r(x) = \{y \in M | d(x,y) < r\}$ and for example 1, the balls can be seen as a ball for 1.a, a box for 1.b, and as diamond for 1.c .
Note, the textbook has a weird notation for Balls:

Fact
There is no metric on $M = \{\text{continues functions } f[a,b] \to \mathbb{R}\}$ with property that

$$f_n(x) \to f(x) \forall x \in [a,b] \Leftrightarrow d_e(f_n, f) \to 0$$

i.e. continuous functions with topological point wise convergence is not a metric space

1b) Basic notation

1b).1 sequence

a **sequence** is function $f : \mathbb{N} \to M$ and is denoted (p_n) or $(p_n)_{n=1}^{\infty}$ an is

1b).2 convergence

a sequence (p_n)**converges to p in M** if

$$\forall \epsilon \; \exists \; \text{N, st. if } n \in \mathbb{N} \text{ and } n > N \Rightarrow d(p_n, p) < \epsilon$$

1b).3 open

$$U \subseteq M \text{ is open if } \forall p \in U \; \exists \gamma > 0 \text{ st. } M_\gamma(p) \subseteq U$$

1b).4 Closed

$$A \subseteq M \text{ is closed if } A^C \text{ is open}$$

Remarks:

- in general, subset of a metric space is closed, open, either or both
- In a metric space, many notions (continuity, compactness, etc) can be characterized in terms of either (1) open sets (2) sequences. Pugh leans more towards using sequences.

1c) Continuity

1c).1 Defintions

1. $\epsilon - \delta$ definition

$$f \text{ is continuous in } M \text{ if } \forall p, q \in M, \exists \delta > 0 \text{ st. } d_n(f(p), f(q)) < \epsilon \text{ wherever } d_m(p,q) < \delta$$

2. $\forall\ U \subseteq N$ open , $f^{-1}(U) = \{x \epsilon M : f(x) \subset U\}$ is open
   (i.e. pre-image of an open set is open)
3. $\forall F \subseteq N$ closed $f^{pre}(F)$ is closed in M
   (image of closed is closed)
4. $\forall (p_n) \to p$ in $M, f(p_n) \to f(p)$
   $\forall$ sequence (p_n) in M converging to a limit p the image sequence $(f(p_n))$ converges to $f(p)$

1c).2 Proofs of equivalence

proof (2) $\to$ (3)
pre-image of compliment = compliment of pre-image

Part of proof (1) to (4)
Assume f satisfies (1).
Consider sequence (p_n) in M converging to p.
Must show that $f(p_n) \to f(p)$
Fix $\epsilon > 0$

$$\exists \delta > 0 \text{ st. } d_N(f(p), f(q)) < \epsilon \text{ if } d_M(p,q) < \delta$$

Also,

$$(p_n) \to p, so \exists N \text{ st. } d_m(p, pn) < \delta\ \forall n \geq N$$

So

$$\forall n \geq N\ d_m(f(p_n), f(p)) < \epsilon$$

Since ϵ arbitrary $f(p_n) \to f(p)$
Proof 4 to 1
Assume (1) fails

$$\exists p \in M, \epsilon > 0 \text{ st. } \forall \delta > 0 \exists q_\delta \text{ st } d_M(p, q_\delta) < \delta$$

$$\text{but } d_n(f(p), f(p_q)) > \epsilon\star$$

consider sequence $(q_{\frac{1}{n}})$ in M
CLearly $q_\delta \to p$ in M
And $f(q_\delta)$ cannot converge to $f(p)$ by $\star$
so not(1) $\Rightarrow$ (1)

1d) Homeomorphism

Definition
M,N metirc space .
A homeomorphism is a continuous bijection with continuous inverse

- M ,N are **homeomorphic** if
  $\exists$ a homeomorphism between them
- If $f : M \to N$ is a Homeomorphisim then it induces a bijection between open sets M and N
  i.e. $U \subseteq M$ open $\Leftrightarrow f(u) \subseteq N$ open

If M metric space and $N \subseteq M$ we can define a metric on N by

$$d_N(x, y) = d_m(x, y) \;\; x, y \in N \subseteq M$$

with this metric N is called a metric subspace

1e) Inheritance principle

If N is a metric subspace of M then

1. $U \subseteq N$ is open $\Leftrightarrow U = N \cap V, V \subseteq M$ open
2. $F \subseteq N$ is closed $\Leftrightarrow F = N \cap G, G \subseteq M$ closed

Example

- Let $M = \mathbb{R}$ and $N = \mathbb{Q}$ consider $S = \{x \in N | -\pi < x < \pi\}$
  then S is **closed** in N (but not in $\mathbb{R}$)
  closed because : $S = n \cap [-\pi, \pi]$
  Proof:
  1. Assume $U = N \cap V$
     V open in M
     $\forall p \in U, \exists M_r p \subseteq V$ so $M_r p \cap N \subseteq V \cap N = U$
     Assume U open in N

     Note:
     only open set is the union of open balls contain in itself

     Thus

$$u = \bigcup \{M_r P | M_r p \subseteq U\}$$

$$\subseteq \bigcup \{M_r p | M_r p \cap N \subseteq U\} = V$$

- M_1, M_2 are metric spaces consider the product space $M = M_1 \cdot M_2$
  different metric spaced for M:
  Say $x, y = (x_1, x_2), (y_1, y_2) \in M$
  Consider the metric spaces d_e, d_{max}, d_{sum} from the 1a example 1

  $$d_e = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$$

  $$d_{max} = max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

  $$d_{sum} = |d_1(x_1, y_1)| + |d_1(x_2, y_2)|$$

  Fact

  $$d_{max} \leq d_e \leq d_{sum} \leq 2d_{max}$$

  Proof

  1. $d_{max} \leq d_E$
  2. $d_e^2 \leq d_{sum}^2$

  So all distances induce the same topology

1f) Cauchy

M is a metric space
(p_n) is a Cauchy Sequence in M if $d(p_n, p_m) \to 0$ as $min\{m, n\} \to \infty$
or (p_n) is a Cauchy Sequence in M if $\forall \epsilon\ \exists N$ st. $\forall n, m \geq N \to d(p_n, p_m) \leq \epsilon$

1g) Completeness

A metric space is **complete** if every Cauchy sequence converges to a limit

1g).1 Completeness Theorem

$\forall M$ subspaces of a Metric space $\exists \hat{M}$ with metric D and function $i : M \to \hat{M}$ st.

1. $\hat{M}$ is complete
2. $\forall x, y, \in M\ d(x, y) = D(i(x), i(y))$
3. $i(M)$ is dense in $\hat{M}$
   (Note that (3) and (2) say that M is a dense metric space of $\hat{M}$) Also $\hat{M}$ is unique up to isometry

Proof: Let C be the set of Cauchy sequences in M st. (p_n) and $q(_n) \in C$
$(p_n) \sim (q_n) if \lim_{n\ \to\infty} d(p_n . q_n) = 0$
Let $\hat{M} = C \backslash \sim$ = equivalent to class in C with respect to $\sim$
We can now write the elements of $\hat{m}$ as $[(q_n)]$

- **Defining Metric and function**

  $$D([p_n, q_n]) = \lim_{n \to \infty} d(p_n, q_n)$$

  $$i(p) = [(p, p, p, ...)] = \text{(mapping to a sequence of itself)}$$

- **Lemma** $p, q, x, y \in M$
$$|d(p,q,) - d(x,y)| \leq d(p,x) + d(q,y)$$
proof:
$$d(p,q) \leq d(p,x) + d(x,y) + d(y,q) \Rightarrow d(p,q) - d(x,y) \leq d(p,x) + d(y,q)$$
$$d(x,y) \leq d(p,x) + d(p,q) + d(y,q) \Rightarrow d(x,y) - d(p,q) \leq d(p,x) + d(y,q)$$

- **Claim:** D is well defined
(limit exists, only depends only equivalence classes)
Fix Cauchy sequences $(p_n)(q_n)$

- **Claim 1:** $(d(p_n, q_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$
By lemma:
$$|d(p,q,) - d(x,y)| \leq d(p,x) + d(q,y)$$
Now assume
$$(p'_n) \sim (p_n)$$
$$(q'_n) \sim (q_n)$$
By lemma:
$$|d(p',q',) - d(p_n,q_n)| \leq d(p'_n,p_n) + d(q'_n,q_n)$$

- **Claim 2:** D is a (suitable) metric
  1. By symmetry $\surd$
  2. positive $\surd$
  3. triangle inequality $\surd$

- **Claim 3:** $\hat{M}$ is complete
Consider Cauchy sequence (P^k) in $\hat{M}$
Goal: Find $Q \in M$ st. $D(P^k, Q) \to 0$ as $k \to \infty$
Note the $P^k = \left[(P_n^k)_{n=1}^{\infty}\right]$

  For each k, I can choose a representative sequence $(p_n^k)_{n=1}^{\infty} \in P_k$ st. $d(p_n^k, p_m^k) \leq \frac{1}{k}$ $\forall n, m$

  Outline:
  Claim: (q_n) is Cauchy, where (q_n) = set of q_is st. $q_i = (p_n)^i$
$$d(q_n, q_m) = d(p_n^n, p_m^m) \leq d(p_n^n, p_l^n) + d(p_l^n, p_l^m) + d(p_l^m, d_n^m)$$
  Note that above, $d(p_n^n, p_l^n) = \frac{1}{n}$, which approaches 0 as $n \to \infty$ and since $(p_l)^k$ is a Cauchy. As a result, it is evident $d(q_n, q_m) < \epsilon$ by taking n and m st. $n, m \geq N$.

- **Define** $\mathbb{Q} = [(q_n)]$

- **Claim:**
$D(P^k, Q) \to 0$ as $k \to \infty$
$$d(p_n^k, q_n) \leq d(p_n^k, p_k^k) + d(p_k^n, p_n^n)$$
where
$$d(p_n^k, p_k^k) < \frac{1}{k} \text{ and } d(p_n^n, p_k^n) = d(q_k, q_n)$$

and we should note that $d(q_k, q_n) \to 0$ as $min(k,n) \to \infty$ So

$$\lim_{k\to\infty} d(p^k, q_n) \to 0$$

$$\Rightarrow \lim_{k\to\infty} D(P^k, Q) = 0$$

1g).2 Remarks

Completeness is not a topological property. i.e. $\exists M, \hat{M}$ homeomorphic st. M is complete and $\hat{M}$ is not complete.

1h) Connectedness

1h).1 Disconnected

Let M be metric space and $A \subseteq M$
A is **disconnected** if :
V1)

- A has proper clopen subset S
- S $\neq \emptyset$
- S $\neq$ A

V2)
$A = \beta_1 \cup \beta_2$ st.

- $\beta_1 \cap \beta = \emptyset$
- $\beta_1, \beta_2 \neq \emptyset$
- both β_1 and β_2 are open

1h).2 Connected

A set is connected if its not disconnected

Remark
A subset of $\mathbb{R}^n$ is connected $\Leftrightarrow$ it is convex

1h).3 Convex

A set is convex if $\forall a, b \in A$, $\exists \theta \in [0,1]$ st. $\theta a - (1-\theta)b \in A$

Fact
If A is connected if $A \subseteq M$ and $f : M \to N$ is continuous, then $f(A)$ is connected (Connectedness is a topological property)

1h).4 Path-connected

If $\forall a, b \in A\ \exists \gamma : [0,1] \to A$ continuous st. $\gamma(0) = a$ and $\gamma(1) = b$ then A is path connected.

Facts:

1. Path-connected $\to$ connected (but not the other direction)
   A great example of the other direction not being true is the "Topologist's sin curve"

2. $f : A \to \{0, 1\}$ where f is continuous, then f is constant $\Leftrightarrow A$ is connected

1i) Rested sequence

M s a metric Space $k_i \subseteq M$

$$... \subseteq k_3 \subseteq k_2 \subseteq k_1$$

Then

$$\cap_{j=1}^{\infty} \neq \emptyset$$

1j) Compactness

1j).1 Sequential compactness

A metric space M is (sequentially) compact if $\forall (p_n) \exists (p_n)_k \to p$ st. $p \in M$

1j).2 Covering compactness

Definitions:

1. **Open Cover**
   An **Open Cover** of M is a collection $U = \{U_\alpha\}_{\alpha \in A}$ of open sets st. $M \subseteq \cup_{\alpha \in A} U_\alpha$

2. **Finite Sub cover**
   A finite subcover is a finite subset of U st. $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ such that

$$M \subseteq \cup_{j=1}^{k} U_j$$

Defining compactness
A metric space M is **(covering) compact** if $\forall$ open covers of M $\exists$ a finite subcover.

Facts:

1. A is compact subset of a complete metric space $\to A$ is closed
   (not closed $\to$ not compact (for subspaces of complete metric spaces))

2. A is compact subset of a complete metric space is bounded

3. For $S \subset \mathbb{R}^n$
   S is compact $\Leftrightarrow$ closed and bounded

4. Continuous image of a compact set is compact
   (Compactness is a topological property)

1j).3 Complete and Totally Bounded

Definition: Totally Bounded A Metric space M is **Totally Bounded** $\forall \epsilon > 0$ M can be covred by a finite union of balls of radius ϵ. i.e. $\exists p_1, \dots, p_l \in M$ st.

$$M \subset \cup_{j=1}^{l} M_\epsilon P_j$$

1j).4 Extreme Value Theorem

If M is compact and $f : M \to \mathbb{R}$ is continuous then

$$\exists p_\star, p^\star \in M \text{ st } \forall q \in M$$

$$f(p_\star) \leq f(q) \text{ and } f(p^\star) \geq f(q)$$

Proof:
Consider (p_n) in M st $f(p_n) \to \inf_{\bar{M}}$
then $\exists (p_{n_k}) \to p$

$$\Rightarrow f(p_{n_k}) \to f(p) = \inf_{\bar{M}} f = \min_{M} f$$

1j).5 Bicompactness

If M is a compact set and $f : M \to N$ is a continuous bijection, then f^{-1} is continuous

Proof:
Assume that f^{-1} is not continuous. If there is a sequence in the image which converges, then the we can, by bijection, there is a sequence in the domain that corresponds to it that does not converge to the same preimage of the limit point of the sequence in the image. Never the less, by the compactness of M, there is subsequence in the domain which converges to a different point then the original sequence in the image. However, since this is a subsequence of this sequence, there is a contradiction.

Lemma
Suppose (p_n) is a sequence in compact space M. If every subsequence converges to p then the entire sequence converges to p.

(Proof by contra positive)

1j).6 Theorem of Compactness

The following definitions are equivalent

1. M is sequentially compact
2. M is covering compact
3. M is complete and totally bounded

Proof: (1) $\Rightarrow$ (2) (2)

1k) Uniform Continuity

Definiton: Uniform Continuity
$f : M \to N$ is uniformly continuous if $\forall \epsilon, \exists \delta$

$$\forall x, y \in M, d(x, y) \Rightarrow d(f(x), f(y))$$

Theorem
$f : M \to N$ is continuous and M is compact, the f is uniformly continuous

Part II
Function Space

2 Convergence of function in C^0

2).1 Sequence of functions

Supposed (f_n) is a sequence of function, st $f_n : [a, b] \to \mathbb{R}$

Definitions

1. **Norm** Norm $|| \cdot ||$ is defined as having a properties:
   (a) $||G|| = 0 \Leftrightarrow G = 0$
   (b) $c||G|| = ||Gc||$ where $c \in \mathbb{R}$ and G is a function
   (c) $||F + G|| \leq ||F|| + ||G||$
2. **Point Wise Convergence**
   $f_n \to f$ **point wise** if $\forall x \in [a, b]$ $f_n(x) \to f(x)$ as $n \to \infty$
3. **Uniform Convergence**
   $f_n \to f$ **uniformly** if $\forall \epsilon$ $\exists N$ st. $\forall x \in [a, b]$ $n \geq N \Rightarrow |f_n(x) - f(x)| \leq \epsilon$
   (Denote uniform convergence as $f_n \rightrightarrows f$)
4. **Uniform Convergence** (Alternate)
   $f_n \rightrightarrows f$ iff $\forall \epsilon \exists N$ st $\forall ||f_n - f|| \leq \epsilon$ when $n \geq N$
   where the norm is defined as $||G|| = \sup_{x \in [a,b]} G(x)$

Thereom: A uniform limit of continuous functions is continuous

Proof:
Assume $f \rightrightarrows f$ $f_n : [a, b] \to \mathbb{R}$. Then fix $x \in [a, b]$ and $\epsilon > 0$.
Choose N large enough st. $||f_n - f|| \leq \frac{\epsilon}{3}$
And find δ st. $|f_n(x) - f_n(x_0)| \leq \frac{\epsilon}{3}$ when $|x - x_0| \leq \delta$

- Claim: $|f(x) - f(x_0)| \leq \epsilon$
  Proof:
  $$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \leq 3\frac{\epsilon}{3} = \epsilon$$

We can imagine that a set of uniformly continuous function is like a tube around a single continuous function which the set converges to.

2a) Notation

- $C_b([a, b], \mathbb{R}) := C_b = \{$ bounded functions : $[a, b] \to \mathbb{R}\}$
- $C^0([a, b]; \mathbb{R}) = C^0 = \{f \in C_b : f$ is continuous $\}$

Note:
We will use

$$||F|| = \sup_{x\in[a,b]} |f(x)|$$

and

$$d(f,g) = ||f-g||$$

2b) Completeness of C_b

C_b is a complete metric space:
Proof:
show completeness by showing that all Cauchy Sequences (f_n) converges. i.e.

$$\forall x \in [a,b] \text{ consider } (f_n(x))_{n=1}^{\infty}$$

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ exists}$$

Claim 1: "d" is metric (exercise)
Claim 2: $f_n \rightrightarrows f$
Proof:
Fix $\epsilon > 0$, $N > 0$ st. $d(f_n, f_m) = ||f_n - f_m|| < \epsilon, \forall n, m \geq N$
Will show that $\forall n \geq N, \forall x \in [a,b]$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \epsilon + |f_n(x) - f(x)|$$

(note that the final equation foes to 0, if $m \geq N$)
And $f_n(x) \to f(x)$ so take the $lim_{n\to\infty}$ on both sides to set $|f_n(x) - f(x)| < \epsilon$

Corollary

1. C^0 is a closed subspace of C^b so C^0 is a complete metric space.
2. Supposed $f_n \in C^0$ satisfies $||f_n|| < M$ and $\sum_{n=1}^{\infty} M_n < \infty$ then

$$F_n = \sum_{k=1}^{\infty} f_k \rightrightarrows \text{ a limit } F = \sum_n = 1^{\infty} f_n$$

(i.e. a series is summable in a sense of uniform convergence)
Also $\sum_{n=1}^{\infty}$ converges in the same sense

Fact:
$f : [a.b] \to \mathbb{R}$ is Riemann integrable iff $\exists z st :$

1. f is continuous at $x \in [a,b]\backslash z$
2. z has measure 0

Theorem: $(f_n) \in C_b$ and f_n is Riemann integrable $\forall n, f_n \rightrightarrows f$ then f is Riemann integrable.

Proof: f_n is R.I. (Riemann Integrable) $\Rightarrow$ $\forall n, \exists z_n$ of measure 0, such that f is continuous on $x \in [a,b]\backslash z_n$ then every f_n is continuous at $x \in [a,b]\backslash z_n$ where $z = \cup_n z_n$.
Then we know that :

1. z has measure 0
2. f is continuous $\forall x \in [a,b] \backslash z$
   (by local version of theorem that uniform limits of continuous functions is continuous)

$\Rightarrow f$ is R.I

Recall
f_n R.I and $f_n \rightrightarrows f$ then f is R.I and

$$\int_a^b f_n dx \to \int_a^b f dx$$

Proof:

$$\left| \int_a^b (f_n - f) dx \right| \leq \int_a^b |f_n - f| \leq \int_a^b ||f_n - f|| dx = (b-a)||f_n - f|| \to 0$$

Corollary: Above Hypothesis implies that

$$\int_a^x f_n(t) dt \to \int_a^x f(t) dt$$

Theorem: f_n is differentiable, and $\forall n$

$$f_n \rightrightarrows f \text{ and } f\_n \rightrightarrows g$$

This is differentiable and $g(x) = f'(x)$ $\forall x \in [a,b]$

Theorem
(f_n) sequence in c^0, f_n is differentiable $\forall x \in (a,b), \forall n$ $f_n \rightrightarrows f$ and $f'_n \rightrightarrows$ some limit g then $f'(x) = g(x)$ $\forall x$

Note: Do not assume f_n is continuously differentiable
(e.g $x^2 sin(\frac{1}{x^2})$ is differentiable but not continuously differentiable)

Proof: Fix $x \in (a,b)$ define the function $t \in [a,b]$

$$\phi_n(x) \quad \begin{cases} \frac{f_n(t)-f(x)}{t-x} & t \neq x \\ f'_n(x) & t = x \end{cases}$$
$$\phi(x) \quad \begin{cases} \frac{f(t)-f(x)}{t-x} & t \neq x \\ g(x) & t = x \end{cases}$$

Hypothesis implies $\phi_n(t) \to \phi(t)$, $\forall t \in [a,b]$
Thus, we have:

1. $\phi_n(t) \to \phi(t)$ $\forall t$ (pointwise converges)

2. ϕ_n is continuous $\forall n$
3. $\phi_n \rightrightarrows \phi$

We know that the first two claims are true, and will now prove the third claim:

It suffices to show that (ϕ_n) s a Cauchy sequence in C^0 since C^0 is complete, and so all Cauchy sequences converge to limit. By the first claim we know that the limit is ϕ and so the limit if the Cauchy sequence must be the same.

Consider:

$$\phi_n(t) - \phi_m(t) = \frac{(f_n - f_m)(t) - (f_n - f_m)(x)}{t - x} = (f_n - f_m)'(z)$$

where we get z using MVT.

Note: $f_n' \rightrightarrows g \Rightarrow (f_n')$ cuachy, with respect to sup norm, implies that $\forall \epsilon \exists N$ st. $n, m \geq N \Rightarrow ||f_n' - f_m'|| < \epsilon$ when $m \leq x|f_n' - f_m'|$. For such n, m it follows that $|\phi_n(t) - \phi_m(t)| < \epsilon$ and since it was arbitrary, $||\phi_n - \phi_m|| < \epsilon$ wherever $m, n \geq N$.

Thus $\phi_n \rightrightarrows \phi$ as claimed, and ϕ is continuous. i.e. :

$$\lim_{t \to x} \phi(t) = \phi(x) = g(x)$$

$$\Rightarrow \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = g(x)$$

Remark:

1. If f_n is continuously differentiable, $f_n \rightrightarrows f$, $f_n' \rightrightarrows g$ then we can argue as follows:

$$\textbf{as } n \to \infty \quad \begin{array}{ccc} f_n(x) & -f_n(a) & = \int_a^x f_n'(t)dt \\ \downarrow & \downarrow & \downarrow \\ f(x) & -f(a) & = \int_a^x g(t)dt \end{array}$$

   Which implies that f is differentiable and $f' = g$

2. Consider the Spaces of functions:

$$\begin{array}{c|c} C_n \sqsupseteq RI \supseteq C^0 & \supseteq (differentiable) \supseteq C^1 \\ \textbf{Closed subsets} & \textbf{Note closed subsets} \end{array}$$

Defining limit of Series Let $F_n = \sum_{k=0}^{n} f_k$ then

$$F_n \rightrightarrows F = \sum_{k=0}^{\infty} f_k$$

So result follows from previous theorem **Corollary:**

1. Supposed $f_k \in C^0$ and $\sum_{k=0}^{\infty} f_k$ convergent uniformly absolutely, then

$$\sum_{k=1}^{\infty} f_k \subset RI \text{ and } \int_a^b \left(\sum_{k=0}^{\infty} f_k\right) dx = \sum_{k=0}^{\infty} \left(\int_a^b f_k dx\right)$$

2. Supposed f_n is differentiable $\forall k$ and that the serious

$$\sum_{k=0}^{\infty} f_k, \sum_{k=0}^{\infty} f_k'$$

   converge uniformly and absolutely, then $(\sum_{k=0}^{\infty} f_k)$ is differentiable and $(\sum_{k=0}^{\infty} f'(k))' = \sum_{k=0}^{\infty} f_k'$

   Proof: follows from previous theorem

3 Power Series

Definition

- A **power series** is an expansion of the form $\sum_{k=0}^{\infty} c_k x^k$ (or $\sum_{k=0}^{\infty} c_k(x-a)^k$ for some a) (for simplicity we will assume $a = 0$)
- A **Radius of Convergence**, R is :

$$R = \frac{1}{\lim_{k\to\infty} \sup |c_k|^{\frac{1}{k}}}$$

Theorem If $r < R$, the power series converges uniformly and absolutely on $[-r, r]$

Proof. By earlier theorems, it satisfies to show that $\sum M_k < \infty$ for $M_k = \max_{[-r,r]} |c_k x^k|$

Fix β st. $r < \beta < R$ which implies that $\frac{1}{\beta} > \frac{1}{R}$.

Since

$$\lim \sup |c_n|^{\frac{1}{n}} = \frac{1}{R}$$

$\exists N$ such that

$$|c_k|^{\frac{1}{n}} < \frac{1}{\beta} \text{ if } k > N$$

This $M_k \leq \left(\frac{r}{\beta}\right)^k$ for $k \geq 1$
And since $\frac{r}{\beta} < 1$ it follows that $\sum m_k$ is summable ☐

Theorem: A power series can integrated and differentiated term by term on its interval of convergence.
i.e. if :

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \text{ on } (-R, R)$$

then

$f'(x) = \sum_{k=0}^{\infty} kc_k x^{k-1}$
$\int_0^x f(x) = \sum_{k=0}^{\infty} \frac{c_k x^{k+1}}{k+1}$

Proof. Let $F_n = \sum_{k=0}^{n} c_k x^k$ and fix $f < R$, we have $F_n \rightrightarrows f$, on $[-\gamma, \gamma]$

We get $F_n' - \sum_{k=0}^{\infty} kc_k x^{k-1}$ by similar results

Therefore it suffices to show that the series $\sum_{k=0}^{\infty} c_k k x^{k-1}$ converges uniformly on $[-r, r]$.

For this, it is sufficient to check that $\sum c_k k x^{k-1}$ has some radius of convergence.
We set:

$$\sum c_k k x^{k-1} = \sum d_k x^k$$

by setting $d_k = c_{k=1}(k+1)$
Need to compare $\sum_{k=0}^{\infty} c_k x^k$ and

$$\sum_{k=0}^{\infty} d_k x^k = \sum_{k=0}^{\infty} (k+1)^{\frac{1}{k}} |c_{k+1}|^{\frac{1}{k}}$$

Then these are equal sicne :
$(k+1)^{\frac{1}{k}} \to 1$ as $k \to \infty$
$|ck+1|^{\frac{1}{k}} = (|c_{k+1}|^{\frac{1}{k}})^{\frac{k}{k+1}}$ □

Note that the proof for integration follows similarly

Corollary Analytical functions are C^{∞} on interval of convergence
Thus, we can define the analytical functions as power series

4 Compactness and Intro to Equicontinuity

Definitions:

- A set in a metric space is **precompact** if its closure is compact
- Let $S \subseteq C^0(M)$ and M is a Metric Space, then S is **equicontinuous** if $\forall \epsilon\ \exists \delta$ st. $|f(x) - f(y)| < \epsilon$ wherever $d_m(x, y) < \delta\ \forall f \in S$
  i.e.:
  - $\forall finS$, f is uniformly continuous
  - given ϵ, $\exists \delta$ that works $\forall f \in S$ and $\forall x, yinM$

Fact
A subset of complete metric space is precompact if and only if totally bounded

4a) Arzela - Ascli Theorem

Supposed $S \subseteq C([0,1])$ and S is equicontinuous and bounded, then S is precompact.
(This is more or less equivalent to: $S \in C^0([a,b])$ Bounded and equicontinuous then S is precompact

Two parts to proof:

1. Consider (f_n), find a sub-sequence (f_{n_k}) such that $f_{n_k} \to$ a limit $f(x)$ $\forall x$ in dense subset of $[0,1]$.
2. Use equicontinuity to improve above convergence to uniform convergence

Proof. 1. Fix a countable dense subset $D \subseteq [0,1]$, $D = \{x_1, x_2, \ldots\}$

   (a) bounded implies that $(f_n(x))_{n=1}^{\infty}$ is bounded sequence in $\mathbb{R}$
   Which implies that $\exists$ sub-sequence $(f_{1_n}(x_1))_{n=1}^{\infty}$ of (f_n) such that $f_{1_n}(x_1) \to \lim f(x)$ (first subsequence)

   (b) Consider $(f_{1_n}(x_2))_{n=1}^{\infty}$ bounded in $\mathbb{R}$
   Which implies that $\exists$ sub-sequence $(f_{2_n}(x_2))_{n=1}^{\infty}$ of (f_{1_n}) such that $f_{2_n}(x_2) \to \lim f(x)$

   (c) Then we keep doing this to find $(f_{k_n})_{n=1}^{\infty}$ sub-sequence of $\left(f_{(k-1)_n}\right)_{n=1}^{\infty}$ and $(f_{k_n}(x_k)) \to \lim f(x_k)$ as $n \to \infty$ $\forall k = 1, \ldots, k$

Claims:

(a) $f_n(x_k) \to f(x_k)$ $\forall k$

Proof. Since the tail of (g_n) is the subsequences $(f_{k_n})_{n=1}^{\infty}$ so

$$\lim_{n\to\infty} g_n(x_k) \to \lim_{n\to\infty} f_n(x_k) = f(x_k)$$

□

(b) $g(m)$ is Cauchy with regards to sup norm.
To prove this we:

   i. fix $\epsilon > 0$
   ii. fix $\delta > 0$ such that

$$|g_m(d) - g_m(d_j)| < \frac{\epsilon}{4}$$

   $\forall g_m$ if $|x - d_j| < \delta$

Thus, we choose point form the countable dense set D

Lemma Claim b)ii) is possible.

Proof. Left as an exercise □

Now fix M large such that

$$|g_n(x_i) - g(x_i)| < \frac{\epsilon}{4} \text{ where } n \geq N, \forall i = 1, \ldots, M$$

Claim:

$$|g_n(y) - g_{n'}(y)| < \epsilon, \forall y \text{ where } n, n' \geq N$$

Proof. Fix $y \in [a,b]$ and find x_i such that $|x_i - y| < \delta$

$f_n(y) - g_{n'}(y)| \leq |g_n(y) - g_n(x_i)| + |g_n(x_i) - f_n(x_i)| + |f_n(x_i) - g_{n'}(x_i)| + |g_{n'}(x_i) - g_{n'}(y)| < \epsilon$

(where the two interior norms are set less than $\frac{\epsilon}{4}$ by setting N large, and the outer norms are set less than $\frac{\epsilon}{4}$ by equicontinuity) □

□

Remarks:

1. Essentially the same proof show that if M, N are compact metric space then a sequence $(f_n) \subseteq C^0(M, N)$ has a convergent subsequence
2. Also a similar argument proves that $S \subseteq C^0([a,b])$ is totally bounded if bounded and equicontinuous

Theoreom: A set $S \subseteq X^0([a,b])$ compact if and only if it is closed, bounded and equicontinuous.

Proof. We note the following:

- We have previously shown that the latter conditions implies the former.
- To show the other direction we want to show that:
  Given $\epsilon > 0$, $\exists \delta$ such that $\forall x, y \in [a,b]$ and $f \in S$, $|f(x) - f(y)| < \epsilon$ (since we know compact $\Rightarrow$ closed and bounded, the only thing left to show is equicontinuity)

Assume S is compact, then it is totaly bounded $\Rightarrow \exists$ a covering of balls $M_{\frac{\epsilon}{3}} f_i \ \forall \epsilon$

Chose δ such that $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}$ when $|x - y| < \delta \ \forall i = 1, \ldots, N$ chosen prior.

Then $\forall x, y$ such that $|x - y| < \delta$ and $f \in S \ \exists f_i$ such that $f \in M_{\frac{\epsilon}{3}} f_i \Rightarrow |f - f_i| = f(f_i, f) < \frac{\epsilon}{3} (\star)$
Then

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) + f_i(y)| + |f_i(y) + f(y)| < \epsilon$$

(since the outer two norms are set less than $\frac{\epsilon}{3}$ by $(\star)$ and the inner norm is set less than $\frac{\epsilon}{3}$ using δ) □

Theorem:
Given some function F, we know that $\exists \epsilon$ such that ???? to the limit

- **Lemma:** Given $(f_n) \subseteq C^0$, f_n differntiable $\forall n, ||f_n|| \leq C \forall n$ then equicontinuous:

  Proof. By MVT :

$$f_n(x) - f_n(y) = (x - y) f_n'(z) \text{ for some } z$$
$$\Rightarrow |f_n(x) - f_n(y)| \leq c|x - y|$$

□