Introductory Computational Mathematics: Modelling

Jonathan Mostovoy - 1002142665
University of Toronto

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1 Assignment 1

1.1 Transforming an ODE to a Dimensionless State

Question. 1.1:

A series resistor-capacitor (RC) circuit with a current source has capacitance C (in farads), resistance R (in ohms), and voltage across the capacitor V (in volts). The circuit is modelled by the differential equation

$$C\frac{dV}{dt} + \frac{V}{R} = i_0 f(t).$$

The forcing function $f(t)$ has *amplitude* i_0 (in amps). Time t is measured in seconds. Scale the dependent and independent variables so that the resulting differential equation is non-dimensional. In other words, introduce a dimensionless time τ (a scaled time) and a dimensionless voltage v (a scaled voltage) so that the differential equation for $v(\tau)$ is dimensionless.

Answer: We first summarize all our units:

$$\begin{aligned}
[C] &= \text{farads} = \frac{s}{\Omega} \\
[R] &= \text{ohms} = \Omega \\
[V] &= [R][i_0] = \text{volts} = \Omega A \\
[f] &= s \\
[i_0] &= \text{amps} = A
\end{aligned}$$

If we therefore set $\tau = \frac{t}{CR}$ and $v = \frac{V}{i_0 R}$, the natural differentials will be: $d\tau = \frac{dt}{RC}$ and $dv = \frac{dV}{i_0 R}$. Thus, in making the proper substitutions to the derivative term:

$$C\frac{dV}{dt} = C\frac{(dv i_0 R)}{(d\tau RC)} = i_0 \frac{dv}{d\tau}$$

Furthermore, making the substitution of $f(t) = f(\tau CR)$, now looking at the full ODE, we see:

$$C\frac{dV}{dt} + \frac{V}{R} = i_0\frac{dv}{d\tau} + \frac{(i_0 R v)}{R} = i_0\frac{dv}{d\tau} + i_0 v = i_0 f(\tau CR)$$

And so dividing by i_0, we see:

$$\frac{dv}{d\tau} = f(\tau CR) - v$$

Which is a dimensionless ODE.

1.2 Buckingham's Pi Theorem

Question. 1.2:

The speed of sound c in a compressible fluid depends on the bulk modulus K with *SI* units $[K] = \text{kg/m/s}^2$ and the medium density ρ with *SI* units $[\rho] = \text{kg/m}^3$. Use Buckingham's Π theorem to determine a relationship between c, K, and ρ. Look-up, and cite, a reference on the Newton-Laplace equation that gives the constant of proportionality in your relation, derived from physical principles.

Answer: We begin by summarizing the units of this problem.

$$[K] = kg/m/s^2, \quad [\rho] = kg/m^2, \quad [c] = m/s$$

Now, we make use of the standard matrix representation involved in applying the Buckingham Π Theorem:

$$\begin{array}{c} \\ kg \\ m \\ s \end{array} \begin{array}{c} \begin{matrix} k & \rho & c \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ 2 & 0 & -1 \end{pmatrix} \end{array} \sim \begin{array}{c} \\ kg \\ m \\ s \end{array} \begin{array}{c} \begin{matrix} k & \rho & c \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & -1 \end{pmatrix} \end{array} \sim \begin{array}{c} \\ kg \\ m \\ s \end{array} \begin{array}{c} \begin{matrix} k & \rho & c \end{matrix} \\ \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

Therefore, since we have a row of zeros in the RREF of our matrix, by Buckingham's Π Theorem, ∃ one dimensionless variable, π_1. Thus, in setting $C = \sqrt{\frac{k}{\rho}}$, we see:

$$[C] = \left[\sqrt{\frac{k}{\rho}}\right] = \sqrt{\frac{[k]}{[\rho]}} = \sqrt{\frac{kg/(ms^2)}{kg/m^3}} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s}$$

Furthermore, from[1], we know that indeed $C = \sqrt{\frac{k}{\rho}}$.

1.3 An Interdependent Economy

Question. 1.3:

In economics, an input-output model describes the interdependencies between different sectors of the economy. Suppose there are n sectors of the economy and that the i^{th} sector produces x_i units of a good. The goods produced by each sector can be inputs for the other sectors. We assume that the j^{th} sector must use $a_{i,j}$ units of the good from sector i to produce one unit of its good. The good from the i^{th} sector has d_i units demanded by consumers. This model takes the form of a linear equation

$$\mathbf{x} = A\mathbf{x} + \boldsymbol{d}, \tag{1}$$

in which $\mathbf{x}$ is the (unknown) vector of units produced, $\boldsymbol{d}$ is the known vector of units demanded, and the matrix A has known entries $a_{i,j}$. Let $M = I - A$. This model is well-posed if 1) M is invertible, so that there is a unique solution and 2) the principal minors of M are all positive, so that the entries of $\boldsymbol{x}$ are non-negative.

1. Explain in words the meaning of Eq. (1).
2. Describe a scenario in which a diagonal entry of A could be non-zero.
3. For $n = 2$, determine the following sensitivities: $S(x_1, d_1)$ and $S(x_1, a_{1,2})$. For which parameter values are these sensitivities very large?

Answer:

1. In this form of the equation, sector i's output is equal to the output of other sectors' output weighted by how much sector j needs of the good produced from sector i to produce one unit of good in its own sector, then plus the demand for i's goods.
2. Let's take an electrical plant for example. Once running, to keep the electricity producing machines running, one needs to supply a fraction of the electricity produced to do so.

[1] https://www.thermaxxjackets.com/newton-laplace-equation-sound-velocity/

3. Since $I - A$ is inevitable, we know that its determinant is non-zero, and hence we have the following:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{pmatrix}^{-1} = \frac{1}{a_{11} - a_{11}a_{22} + a_{12}a_{21} + a_{22} - 1}\begin{pmatrix} a_{22}-1 & -a_{12} \\ -a_{21} & a_{11}-1 \end{pmatrix}$$

$$:= \frac{1}{\det^*(M)}\begin{pmatrix} a_{22}-1 & -a_{12} \\ -a_{21} & a_{11}-1 \end{pmatrix}$$

(**IMPORTANT NOTE:** In the following computations, we use $\det^*(M)$ to denote $(-1)\cdot\det^*(M)$.)

We therefore have:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M^{-1}\boldsymbol{d} = \frac{1}{\det^*(M)}\begin{pmatrix} a_{22}-1 & -a_{12} \\ -a_{21} & a_{11}-1 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \frac{1}{\det^*(M)}\begin{pmatrix} (a_{22}-1)d_1 - a_{12}d_2 \\ -a_{21}d_1 + (a_{11}-1)d_2 \end{pmatrix}$$

We may now calculate $S(x_1, d_1)$ and $S(x_1, a_{12})$ explicitly:

$$S(x_1, d_1) = \frac{\partial x_1}{\partial d_1}\left(\frac{d_1}{x_1}\right) = \frac{(a_{22}-1)}{\det^*(M)}\frac{\det^*(M)d_1}{((a_{22}-1)d_1 - a_{12}d_2} = \frac{(a_{22}-1)d_1}{(a_{22}-1)d_1 - a_{12}d_2}$$

$$\begin{aligned}
S(x_1, a_{12}) = \frac{\partial x_1}{\partial a_{12}}\left(\frac{a_{12}}{x_1}\right) &= \left(\frac{d_1(a_{22}-1)}{-\det^*(M)^2}\frac{\partial \det^*(M)}{\partial a_{12}} - \left(\frac{d_2}{\det^*(M)} + \frac{a_{12}d_2}{-\det^*(M)^2}\frac{\partial \det^*(M)}{\partial a_{12}}\right)\right)\left(\frac{a_{12}}{x_1}\right) \\
&= \left(\frac{-d_1(a_{22}-1)a_{21}}{\det^*(M)^2} - \left(\frac{d_2}{\det^*(M)} - \frac{a_{12}d_2a_{21}}{\det^*(M)^2}\right)\right)\left(\frac{a_{12}\det^*(M)}{(a_{22}-1)d_1 - a_{12}d_2}\right) \\
&= \left(\frac{-d_1(a_{22}-1)a_{21}}{\det^*(M)} - \left(d_2 - \frac{a_{12}d_2a_{21}}{\det^*(M)}\right)\right)\left(\frac{a_{12}}{(a_{22}-1)d_1 - a_{12}d_2}\right) \\
&= \left(\frac{-d_1(a_{22}-1)a_{21}}{\det^*(M)} - \left(\frac{d_2\det^*(M)}{\det^*(M)} - \frac{a_{12}d_2a_{21}}{\det^*(M)}\right)\right)\left(\frac{a_{12}}{(a_{22}-1)d_1 - a_{12}d_2}\right) \\
&= \left(\frac{a_{12}d_2a_{21} - d_2\det^*(M) - d_1(a_{22}-1)a_{21}}{\det^*(M)}\right)\left(\frac{a_{12}}{(a_{22}-1)d_1 - a_{12}d_2}\right) \\
&= \left(\frac{-a_{21}\big((a_{22}-1)d_1 - a_{12}d_2\big) - d_2\det^*(M)}{\det^*(M)}\right)\left(\frac{a_{12}}{(a_{22}-1)d_1 - a_{12}d_2}\right) \\
&= \left(\frac{-a_{21}a_{12}}{\det^*(M)}\right) - \left(\frac{d_2a_{12}\det^*(M)}{\det^*(M)\big((a_{22}-1)d_1 - a_{12}d_2\big)}\right) \\
&= -\left(\frac{a_{21}a_{12}}{\det^*(M)} + \frac{d_2a_{12}}{(a_{22}-1)d_1 - a_{12}d_2}\right)
\end{aligned}$$

And so when $(a_{22}-1)d_1 \to a_{12}d_2$ and $\det^*(M) \to 0$, the given sensitivities will be extremely sensitive.

1.4 Numerical Eigenvalue Computation

Question. 1.4:

Let A be a real, square, symmetric matrix. Use the method of Lagrange multipliers to find the stationary points of the function $R(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$. Explain how this result could be used to (numerically) compute the eigenvalues of the matrix A.

Answer: We begin by providing an elementary result from Matrix Calculus and an easy Lemma to make the calculations for this problem painless.

Lemma. 1.1: Matrix Calculus Lemmas

If $\mathbf{x}$ is an $(n \times 1)$ variable vector, and A an $(m \times n)$ constant matrix. Then we have the following general result:

$$\frac{\partial \mathbf{x}^T A \mathbf{x}}{\partial \mathbf{x}} = (A + A^T)\mathbf{x}$$

Note: we use the "Denominator layout", for this Question, i.e.; by $\mathbf{y}^T$ and $\mathbf{x}$.

Corollary. 1.1: Matrix Calculus Lemmas

IIf $\mathbf{x}$ is an $(n \times 1)$ variable vector, and A an $(m \times n)$ constant symmetric $(A = A^T)$ matrix. symmetric. Then we have the following general results:

$$\frac{\partial \mathbf{x}^T A \mathbf{x}}{\partial \mathbf{x}} = 2A\mathbf{x}$$

$$\frac{\partial \mathbf{x}^T \mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{x}$$

Proof. Since $A = A^T$, the first equation is immediate from Lem 1.1. The second is achieved by taking $A = I$. □

We now shift focus to the question at hand. We set up our Lagrangian as follows:

$$\mathscr{L}(\mathbf{x}, \lambda) = \mathbf{x}^T A \mathbf{x} - \lambda(\mathbf{x}^T \mathbf{x} - 1)$$

And so:

$$\mathscr{L}_{\mathbf{x}} = 2A\mathbf{x} - \lambda\mathbf{x} \quad \text{by Cor. 1.1,} \quad \text{and} \quad \mathscr{L}_{\lambda} = \mathbf{x}^T\mathbf{x} - 1$$

And so setting our derivatives equal to zero, we find: $A\mathbf{x} = \lambda\mathbf{x}$, and $\mathbf{x}^T\mathbf{x} = 1$. We recognize the former to be the standard equation for eigenvalues (λ), and eigenvectors $(\mathbf{x})$ of the matrix A. Combining these two equations yields:

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T \lambda \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x} = \lambda \quad \text{I.e.} \quad \mathbf{x}^T A \mathbf{x} = \lambda$$

What we have done now is solved for the maximal value of our function subject to the given constraint; namely, the maximal value is achieved by the eigenvector associated to the largest eigenvalue of A.

In terms of numerically solving for the eigenvalues of the matrix A, the following would be a possibility. Firstly, let us assume that we have m unique eigenvalues, $\lambda_1, \ldots, \lambda_m$, where $\lambda_1 > \cdots > \lambda_m$. We maximize $\mathbf{x}^T A \mathbf{x}$ subject to $\mathbf{x}^T\mathbf{x} = 1$ to find the first Eigenvalue, λ_1 (for example, via Lanczos' Algorithm). Let us call the value which maximizes the above as $\hat{\mathbf{x}} = \mathbf{x}_1$. Next, we look for a solution to the following problem: maximize $\mathbf{x}^T A \mathbf{x}$ subject to $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{x}_1 = 0$. Since all eigenvectors are orthogonal to one other, the added constraint of $\mathbf{x}^T\mathbf{x}_1 = 0$ will yield the a solution $\mathbf{x}_2$, the eigenvector associated to the maximal value of $\mathbf{x}A\mathbf{x}^T$ which is preciesly an eigenvalue of A (λ_1 or λ_2). Carrying out this method while adding upon constrains (namely $\mathbf{x}^T\mathbf{x}_1 = \cdots = \mathbf{x}^T\mathbf{x}_k = 0$ at the kth iteration of this method will eventually yield to explicitly discovering every eigenvalue of A.

1.5 Constraint Optimization: Lagrange Multipliers

Question. 1.5:

A shipping company has the capacity to move 100 tons/day by air. The company charges $250/ton for air freight. Besides the weight constraint, the company can only move 50,000 ft^3 of cargo per day because of limited volume of aircraft storage compartments. The following amounts of cargo are available for shipping each day:

Cargo	Weight (tons)	Volume (ft^3/ton)
1	30	550
2	40	800
3	50	400

(a) Determine how many tons of each cargo should be shipped by air each day in order to maximize revenue. Use the five-step method, and model as a constrained optimization problem. Solve using Lagrange multipliers.

(b) Calculate the shadow prices for each constraint, and interpret their meaning.

Answer:
SUPER SUPER IMPORTANT NOTICE: Below you'll find **two solutions**, the first is the proper interpretation, which I used MATLAB to help solve as provided in the hint. The second is how I first incorrectly interpreted the question, namely in that you could only choose discrete packages of Cargo's 1,2, or 3.

1.5.1 Correct Interpretation

(a) We answer this problem via the five-step method.

1. The first step is to ask the question. Our question here is how can this shipping company maximize its income while adhering to volume, tonnage, and discrete shipping quantities.
2. Our modelling approach will be via constraint optimization.
3. We now make explicit the problem at hand. Let x, y, z denote the amount of sent Cargo within the Cargo's 1,2, and 3 respectively. Our objective therefore will be to maximize income, subject to several constraints. Explicitly:

$$\begin{aligned} \text{maximize: } 250(x+y+z) \quad & \text{subject to:} \\ & (1) \quad x+y+z \leq 100 \\ & (2) \quad 550x+800y+400z \leq 50000 \\ & (3) \quad x \leq 30 \\ & (4) \quad y \leq 40 \\ & (5) \quad z \leq 50 \end{aligned}$$

4. We now solve the model via Lagrange Multipliers. We introduce the Lagrangian as:

$$\begin{aligned} \mathscr{L}(x,y,z,\lambda_1,\ldots,\lambda_5) = \Big\{ 250(x+y+z) &- \lambda_1\big(x-30\big) \\ &- \lambda_2\big(y-40\big) \\ &- \lambda_3\big(z-50\big) \\ &- \lambda_4\big(x+y+z-100\big) \\ &- \lambda_5\big(550x+800y+400z-50000\big)\Big\} \end{aligned}$$

Where we allow all the λ's to non-zero or zero. As such, by hand we would have $2^5 = 32$ cases to check, and so we make use of the provided MATLAB function to easy our computations. In doing so, we find the optimal solution as:

$$(\hat{x}, \hat{y}, \hat{z}) = (30, 16.875, 50), \quad \Rightarrow \quad f(\hat{x}, \hat{y}, \hat{z}) = \$24218.75$$

5. After performing our optimization, we see that given the constraints, the most optimal strategy is to ship a full container of Cargo 1 & 3, and 16.875/40 tons of Cargo 2. This strategy will yield an income of \$24218.75.

(b) We first note that in our solution, only λ_1, λ_3, and λ_5 will be active since $\hat{y} < 40$ and $\hat{x}+\hat{y}+\hat{z} < 100$. Therefore, $\lambda_2 = \lambda_4 = 0$; I.e., a small change in the values of 40 and 100 respectively will have no impact on our optimal solution. In taking derivatives with respect to x, z, and λ_5, and setting them equal to zero and plugging in $(\hat{x}, \hat{y}, \hat{z})$, we find that $\lambda_1 = \$78.125/\text{ton}$, $\lambda_3 = \$125/\text{ton}$, and $\lambda_4 = \$0.3125/\text{ton}$. I.e., when our constraints our flexed by 1 ton, we can expect our income to change in the same direction by the given values above.

1.5.2 A Secondary Interpretation

(a) We answer this problem via the five-step method.

1. The first step is to ask the question. Our question here is how can this shipping company maximize its income while adhering to volume, tonnage, and discrete shipping quantities.
2. Our modelling approach will be via constraint optimization.
3. We now make explicit the problem at hand. Let x, y, z denote the discrete (non-negative) quantities of cargo 1,2,3 respectively. Our objective therefore will be to maximize income, subject to several constraints. Explicitly:

$$\begin{aligned}
\text{maximize: } 250(30x + 40y + 50z) \quad & \text{subject to:} \\
& (1) \quad 30x + 40y + 50z \leq 100 \\
& (2) \quad 30 \cdot 550x + 40 \cdot 800y + 50 \cdot 400z \leq 50000 \\
& (3) \quad (x-3)(x-2)(x-1)(x) = 0 \\
& (4) \quad (y-1)(y) = 0 \\
& (5) \quad (z-2)(z-1)(z) = 0
\end{aligned}$$

Our income equation is immediate from the Inequality (1) is immediate from the restriction that gross tonnage must be less than 100tons. Inequality (2) comes from the fact that this company can only move a maximum of 50000 ft^3, and cargo 1, 2, and 3's volume is 16500, 32000, and 20000 tons respectively. To restrict our possible values of x, y, z to non-negative integers, we introduce equations (3), (4), and (5). Intuitively, for (3), the equation $(x-3)(x-2)(x-1)(x) = 0$ is equivalent to $x \in \{0, 1, 2, 3\}$ [and similarly for (4) and (5)]. The fact that x, y, z can only assume values from $\{0, 1, 2, 3\}, \{0, 1\}$, and $\{0, 1, 2\}$ respectively is because $30 \cdot (x|_{x\in\mathbb{N}, x>3}) > 100$, $32000 \cdot (y|_{y\in\mathbb{N}, y>1}) > 50000$, and $50 \cdot (z|_{z\in\mathbb{N}, z>2}) > 100$.

4. We now solve the model via Lagrange Multipliers. We introduce the Lagrangian as:

$$\begin{aligned}
\mathscr{L}(x, y, z, \lambda_1, \ldots, \lambda_5) = \Big\{ 250(30x + 40y + 50z) & - \lambda_1\big((x-3)(x-2)(x-1)(x)\big) \\
& - \lambda_2\big((y-1)(y)\big) \\
& - \lambda_3\big((z-2)(z-1)(z)\big) \\
& - \mathbb{1}_1\lambda_4\big(30x + 40y + 50z - 100\big) \\
& - \mathbb{1}_2\lambda_5\big(30 \cdot 550x + 40 \cdot 800y + 50 \cdot 400z - 50000\big) \Big\}
\end{aligned}$$

Where $\mathbb{1}_1, \mathbb{1}_2$ attain the values 0 or 1 depending on if we are looking for an interior or boundary (respectively) solution with respect to the given inequalities. We now may actually cheat slightly by only taking $\mathscr{L}$'s derivatives with respect to $\lambda_1, \ldots, \lambda_5$ since simply checking these cases will quickly yield our answer.

$$\mathscr{L}_{\lambda_1} = 0 \equiv \text{Restraint (3)}, \quad \mathscr{L}_{\lambda_2} = 0 \equiv \text{Restraint (4)}, \quad \mathscr{L}_{\lambda_3} = 0 \equiv \text{Restraint (5)}$$
$$\mathscr{L}_{\lambda_4} = 0 \equiv \text{Restraint (1)}, \quad \mathscr{L}_{\lambda_5} = 0 \equiv \text{Restraint (2)}$$

(We provide $\mathscr{L}_x, \mathscr{L}_y$, and $\mathscr{L}_z$ part (b).)

And so by (3), (4), (5). The only possible values for (x, y, z) are:

$$(x, y, z) \in \Big\{(s, u, v) \;\Big|\; s \in \{0, 1, 2, 3\}, u \in \{0, 1\}, v \in \{0, 1, 2\}\Big\}$$

However, we may additionally reduce the possible values of (x, y, z) down to only a few since otherwise (1) & (2) will not be satisfied. Namely now:

$$(x, y, z) \in \Big\{(0, 0, 0), (1, 0, 0), (2, 0, 0), (3, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 0), (0, 0, 1), (0, 0, 2)\Big\}$$

Furthermore, one can immediately reduce the number of possible values by only including the maximal single-active variable conditions, and so now:

$$(x, y, z) \in \Big\{(3, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 0), (0, 0, 2)\Big\}$$

Finding the answer now is a trivial task of plugging in the 5 possible values into the shipping company's income function, $\pi = 250(30x + 40y + 50z)$. One finds that:

$$\pi|_{(x,y,z)=(3,0,0)} = 22500, \quad \pi|_{(x,y,z)=(1,1,0)} = 17500, \quad \pi|_{(x,y,z)=(1,0,1)} = 20000$$
$$\pi|_{(x,y,z)=(0,1,0)} = 10000, \quad \pi|_{(x,y,z)=(0,0,2)} = 25000$$

5. With this information, we are able to conclude that when subject to the given constraints, the income maximizing strategy for the shipping company is to ship 2 "Cargo 3"'s, which yields an income $\pi = 25000$, with a Weight $= 100\text{tons} \leq 100\text{tons}$, and Volume $= 40000\text{ft}^3 \leq 50000\text{ft}^3$.

(b) We begin by calculating $\mathscr{L}_x, \mathscr{L}_y$, and $\mathscr{L}_z$:

$$\mathscr{L}_x = 7500 - \lambda_1(4x^3 - 18x^2 + 22x - 6) - 30\mathbb{1}_1\lambda_4 - 16500\mathbb{1}_2\lambda_5$$
$$\mathscr{L}_y = 10000 - \lambda_2(2y - 1) - 40\mathbb{1}_1\lambda_4 - 32000\mathbb{1}_2\lambda_5$$
$$\mathscr{L}_z = 12500 - \lambda_3(3z^2 - 6z + 2) - 50\mathbb{1}_2\lambda_4 - 20000\mathbb{1}_2\lambda_5$$

Since Constraint (2) is inactive in our solution, we have $\mathbb{1}_2 = 0$, and so λ_5 does not exist. Furthermore, by plugging in our solution of $(x, y, z) = (0, 0, 2)$, our equations reduce down to:

$$\mathscr{L}_x = 7500 - \lambda_1(-6) - 30\mathbb{1}_1\lambda_4$$
$$\mathscr{L}_y = 10000 - \lambda_2(-1) - 40\mathbb{1}_1\lambda_4$$
$$\mathscr{L}_z = 12500 - \lambda_3(3(2)^2 - 6(2) + 2) - 50\mathbb{1}_2\lambda_4 = 12500 - \lambda_3(2) - 50\mathbb{1}_2\lambda_4$$

However, since we already found a discrete solution subject to our constraints, we may equivalently state the Lagrangian without the discrete properties inherently as a constraint within, and simply limit our solutions to that of discrete cases. Namely:

$$\mathscr{L} = 250(30x + 40y + 50z) - \lambda_4(30x + 40y_50z - 100)$$

which would yield:

$$\mathscr{L}_z = 250(50z) - 50\lambda_4$$

And setting $\mathscr{L}_z = 0$, we finally have: $\lambda_4 = 250$ (as it would've also yielded had we taken $\mathscr{L}_x$ or $\mathscr{L}_y$). Intuitively therefore, a unit increase in the constraint ≤ 100 equates to \$250 increase in income. Furthermore, since λ_5 does not exist, intuitively this means that a unit increase in the constraint ≤ 50000 has no effect on income.