A Review of Analysis

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Contents

1 Some Basic Definitions

1.1 Covers

Definition. 1.1: Covers Subcovers and Open Subcovers

Let $U = \{U_i\}_{i \in I}$ be a collection of subsets of M, I.e., $U_i \subseteq M \; \forall i \in I$, I is an indexing set. We say U covers $A \subseteq M$ if $A \subseteq \cup_{i \in I} U_i$. We also equivalently call $U = \{U_i\}_{i \in I}$ a covering of A. If both $U = \{U_i\}_{i \in I}$ and $V\{V_j\}_{j \in J}$ cover A, and $V \subseteq U$ in the sense that $\forall V_j \in V, V_j \in U$, then we say U reduces to V, and V is a subcovering of A. If all the sets in a covering U of A are open then U is an open covering of A. If every open covering of A reduces to a finite subcovering of A then we say that A is covering compact.

1.2 Measure Theory

Definition. 1.2: Lebesgue Outer Measure

The Lebesgue outer measure of a set $A \subset \mathbb{R}$ is a function $m^* : A \to \mathbb{R}^+$, I.e.,

$$m^*(A) := \inf \left\{ \sum_k |I_k| : \{I_k\} \text{ is a covering of } A \text{ by open intervals} \right\}$$

If every series $\sum_k |I_k|$ diverges, then we say definition $m^*(A) = \infty$.

Theorem. 1.1: Properties of Measurability

1. The outer measure of the empty set is 0, I.e., $m^*(\emptyset) = 0$.
2. If $A \subset B$, then $m^*(A) \leq m^*(B)$.
3. If $A = \cup_{j \in J} A_j$, then $m^* \leq \sum_{j \in J} m^*(A_j)$.

1. *Proof.* Since all sets $S \in \mathbb{R}$ cover $\emptyset$, we take $I = \{0\}$ which has length 0. □
2. *Proof.* Let us say V covers B, and hence also covers A. As such, if U is a subcover of V s.t. it covers A, then this fact is obvious. □
3. See Pugh for proof.

2 Vector Spaces

Definition. 2.1: Vector Space

A Vector Space, V is a collection of elements for which we have addition and multiplication by a number, real or complex, defined pointwise. i.e., if $x, y \in V \implies \alpha x + \beta y \in V$.

Some examples of vector spaces are ($\Omega \subset^{open} \mathbb{R}^n$):

1. $\mathbb{R}^n$ where $x + y = (x_1 + y_1, \ldots, x_n + y_n)$ and $\alpha x := (\alpha x_1, \ldots, \alpha x_n)$.
2. $C^k(\Omega) := \{f : \mathbb{R}^n \to \mathbb{C} | \ \partial^\alpha f$ exists and is continuous $\forall \alpha \leq k\}$ where $(f+g)(x) := f(x)+g(x)$ and $(\alpha f)(x) := \alpha f(x) \ \forall f, g \in C^k(\Omega)$.
3. $L^p(\Omega) = \{f : \mathbb{R}^n \to \mathbb{C} : \int |f|^p < \infty\}$ where $(f + g)(x) := f(x) + g(x)$ and $(\alpha f)(x) := \alpha f(x)$ $\forall f, g \in L^p(\Omega)$.
4. $\mathcal{S}(\Omega) : C^\infty$ functions which decay together with all their derivatives faster than $|x|^{-n} \ \forall n \in \mathbb{N}$ where $(f + g)(x) := f(x) + g(x)$ and $(\alpha f)(x) := \alpha f(x) \ \forall f, g \in \mathcal{S}(\Omega)$.

2.1 Locally Convex Spaces

We skip the formal definitions of a Locally Convex Space due to its lack of direct application in most of the courses to be studied at both graduate and undergraduate classes. However, it is important to note that this structure is induced by what is to be soon defined a Normed Space and a Inner Product Space, and once induced, a Locally Convex Space induces a either a Topological or Vector Space (depending on what we were originally working with).

2.2 Normed Spaces

Definition. 2.2: Normed Space & Norm

We may define a norm, $||v||$ on a Vector Space V if it satisfies the following properties $\forall u, v \in V$:

1. $||\alpha u|| = |\alpha| ||u||$
2. $||u + v|| \leq ||u|| + ||v||$
3. $||v|| = 0 \iff v = 0$

A vector space V together with a norm, $||v||$ is called a normed space, denoted $(V, ||v||)$.

Thus, from a norm, we get an idea of size and closeness between two elements of V. This is where neighborhoods arise and hence what topology is interested in.

2.2.1 Sequences

Definition. 2.3: Convergence

We say a sequence of elements of a normed (vector) space, $(v_n)_{n=1}^{\infty} \in V\ \forall n$ converges to v (written $v_n \to v$ or $\lim_{n\to\infty} v_n = v$) $\iff ||v_n - v|| \to 0$.
Formally, $v_n \to v$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $||v_n - v|| < \epsilon$.

Some examples of Convergent Sequences are:

1. $x_n = \frac{1}{n} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ which converges to 0.
2. $y_n = (1 + \frac{a}{n})^n \to e^a$.
3. $\zeta_n(2) = \sum_{r=1}^{n} r^{-2} \to \frac{\pi^2}{6}$

Definition. 2.4: Cauchy

We say a sequence $(v_n)_{n=1}^{\infty} \in V$ is a Cauchy Sequence if $||v_m - v_n|| \to 0$.
Formally, v_n is Cauchy if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$, $||v_m - v_n|| < \epsilon$.

Some Examples of Cauchy Sequences are:

1. $x_n = \frac{1}{x}$
2. $f_n = \frac{\sin(x)}{n}$

Definition. 2.5: Complete

We say a space V is complete if every Cauchy Sequence, $v_n \in V$ converges to a point $v \in V$.

2.2.2 Banach Spaces

Definition. 2.6: Banach Space

A Banach Space is a complete normed space.

Some examples of Banach Spaces are:

1. $(R^n, ||x||)$, where $||x|| = \sqrt{\sum x_j^2}$
2. $(C^k(\Omega), ||f||_{C^k})$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and if $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N}, \dots, \mathbb{N})$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$
$$||f||_{C^k} = \max_{|\alpha| \leq k} \left(\sup_{x \in \Omega} \left| \frac{\partial^{\alpha} f(x)}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \right| \right)$$
3. $(L^p(\mathbb{R}^n), ||f||_{L^p})$, where
$$||f||_p = \left(\int |f|^p \right)^{\frac{1}{p}}$$

2.2.3 L^p Spaces

Let $\Omega \subset^{open} \mathbb{R}^n$, and dx denote the Lebesgue measure. We thus define the L^p Spaces for $1 \leq p < \infty$:

$$L^p(\Omega) := \left\{ f : \Omega \to \mathbb{C} \middle| f \text{ is measurable, and } \int |f|^p dx < \infty \right\}$$

We may thus make the realization that: $f \in L^p(\Omega) \iff |f|^p \in L^1(\Omega)$.

We are thus left to define $L^\infty(\Omega)$ as:

$$L^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{C} \middle| f \text{ is measurable, and } \left(\inf \left\{ \sup |g| \;\middle|\; g = f \text{ almost everywhere} \right\} \right) < \infty \right\}$$

As such, we have now defined another set of Vector Spaces: $\{L^p | p \in [1, \infty]\}$.

Theorem. 2.1: Hölder's Inequality

If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and $f \in L^p$, $g \in L^q$, then $fg \in L^r$ and:

$$||fg||_{L^r} \leq ||f||_{L^p} ||g||_{L^q}$$

Proof. Forthcoming □

Corollary. 2.1: Minkowski's Inequality

If $f, g \in L^p$, then:

$$||f + g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p}$$

Proof. Forthcoming... □

Corollary. 2.2: Less General Hölder's Inequality

If $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p$, $g \in L^q$, then:

$$\int |fg| dx \leq ||f||_{L^p} + ||g||_{L^q}$$

Proof. Forthcoming... □

Corollary. 2.3: Jensen's Inequality

Let φ be a convex function on $[a, b]$ (i.e., φ satisfies $\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y)$ $\forall \alpha \in [0, 1], x, y \in [a, b]$), and $p_k \in \mathbb{R}^+ \forall k \in \{1, \ldots, n\}$ which also satisfy $\sum_{k=1}^{n} p_k = 1$. Then $\forall x_k \in [a, b]$:

$$\varphi\left(\sum_{k=1}^{n} p_k x_k \right) \leq \sum_{k=1}^{n} p_k \varphi(x_k)$$

Proof. Forthcoming... □

2.3 Inner Product Spaces

Definition. 2.7: Inner Product Space & Inner Product

A Vector space V together with a inner product, $\langle u, w\rangle$ is an inner product pace, denoted $(V, \langle u, w\rangle)$. $\langle u, w\rangle$ is an inner product in said (vector) space V if it satisfies the following $\forall u, w, z \in V$:

1. $\langle \alpha u, w\rangle = \bar{\alpha}\langle u, w\rangle$ and $\langle u + z, w\rangle = \langle u, w\rangle + \langle z, w\rangle$
2. $\langle w, u\rangle = \overline{\langle u, w\rangle}$
3. $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \iff u = 0$

Some examples of inner product spaces are:

1. $(\mathbb{R}^n, \langle x, y\rangle)$, where $\langle x, y\rangle = \sum x_i y_i$
2. $(C_b^k(\Omega), \langle f, g\rangle)$ where $\langle f, g\rangle = \int_\Omega \bar{f}g$

2.3.1 Hilbert Spaces

Definition. 2.8: Hilbert Space

A Hilbert Space is a complete inner product space.

Example (1) from 1.3 is an example of a Hilbert Space.

2.4 Sobolev Spaces

Our motivation for introducing the next space is because oftentimes we need our L^p Space of interest to have additional structure to measure smoothness similar to how we did so in C^k. We do so only for $p = 2$, i.e. for the space $L^2(\Omega)$. This is the most useful space among the L^p Space as it has the inner product:

$$\langle f, g\rangle := \int \bar{f} g dx$$

Hence it satisfies the conditions to be classified a Hilbert Space. Another advantage of the L^2 space is that the Fourier transform leaves it invariant (i.e. $f \in L^2 \implies \bar{f} \in L^2$).

Definition. 2.9: Sobolev Space

A Sobolev Space of degree $s \in \mathbb{N} \cup \{0\}$, denoted $H^s(\mathbb{R}^n)$ is defined as:

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) | \ \partial^\alpha f \in L^2 \ \forall \alpha, |\alpha| \leq s \right\}$$

Where if $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we define:

$$\partial^\alpha \int f\varphi := (-1)^{|\alpha|} \int f \partial^\alpha \varphi$$

To define a suitable Hilbert Space, we may equip our Sobolev Space with the inner product:

$$\langle f, g \rangle_{H^s} := \sum_{|\alpha| \leq s} \langle \partial^\alpha f, \partial^\alpha g \rangle$$

Another way to define a $H^s(\mathbb{R}^n)$ is with the Fourier Transform, i.e.,

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) | \ \sqrt{1 + |k|^2} \hat{f}(k) \in L^2(\mathbb{R}^n) \right\}$$

The definition involving the Fourier Transform above has the advantage that it makes sense for an arbitrary $s \in \mathbb{R}$.

3 Operator Theory

Definition. 3.1: Operator

An operator is a mapping from one (vector) space to another $A : U \to V$.

Some examples of operators are:

1. $A = \mathbb{1}$ (the identity map) which is defined as $F(u) = u$.
2. $A = \Delta$ (the laplacian on H^2 or C^2) which is defined as $F(f) = \sum (\frac{\partial f}{\partial x_i})^2$.
3. $A = \mathscr{F}$ (the Fourier transform) which is defined as $F(u) = (2\pi)^{-\frac{n}{2}} \int e^{-ikx} u(x) dx$.

Linear Operators are Operators with the property that $\forall u_1, u_2 \in U, \alpha, \beta \in \mathbb{C}$, $A(\alpha u_1, \beta u_2) = \alpha A u_1 + \beta A u_2$. We thus may naturally define both the domain and ranges for a linear operator as follows:

Definition. 3.2: Domain & Range

If $A : U \to V$, then the Domain and Range of A are defined respectively as:

$$\mathcal{D}(A) := U$$
$$\text{Ran}(A) := \left\{ Au | u \in U \right\}$$

3.1 Bounded Operators

We now introduce the notion of a Norm for an operator as follows:

Definition. 3.3: Operator Norm

If $A : U \to V$, the norm of A, denoted $||A||$ is defined as:

$$||A|| := ||A||_{U\to V} = \sup_{||u||_U=1} ||Au||_V$$

If $U = V$ and $||A|| < \infty$, the we say A is a bounded operator.

3.2 Inverse Operators

Definition. 3.4: Invertible Operator

We say an operator $A : U \to V$ is invertible $\iff$ $\exists$ an operator $A^{-1} : V \to U$ such that $A^{-1}A = \mathbb{1}_U$ and $AA^{-1} = \mathbb{1}_V$.

Theorem. 3.1: Existence of Inverses & Neumann Series

Assume an operator $A : U \to V$ is invertible and an operator $B : V \to V$ is bounded with the norm satisfying the inequality:

$$||B|| \leq ||A^{-1}||^{-1}$$

Then the operator $A + B$ (defined on the domain of A) is invertible. Moreover, its inverse is given by the absolutely convergent series:

$$(A + B)^{-1} = \sum_{n=0}^{\infty} A^{-1}(-BA^{-1})^n$$

Which is called the Neumann series (for $A + B$).

Proof. Forthcoming... □

4 Fourier Analysis

5 Problems

5.1 Question 1

Show the local existence property for the Hartree equation:

$$i\frac{\partial u}{\partial t} = -\Delta u + (v * |u|^2)u, \quad u|_{t=0} = u_0$$

In the Sobolev spaces $H^s(\mathbb{R}^n)$, $s \geq 0$.

Proof. □

5.2 Question 2

Prove the local existence property for the nonlinear Schrödinger equation:

$$i\frac{\partial u}{\partial t} = -\Delta u + \lambda |u|^{p-1}u, \quad u|_{t=0} = u_0$$

in the Sobolev spaces $H^s(\mathbb{R}^n)$, with $s > n/2$ and p, an odd integer. (Hint: Use Sobolev embedding theorems, e.g. $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ for $s > n/2$, so that products of H^s functions are again H^s functions, i.e. H^s is an algebra.)

Proof. □

5.3 Question 3

Compute $dF(u)$ for $F : \mathbb{R}^n \to \mathbb{R}^m$, and for:

$$F(u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)$$

Proof. □

5.4 Question 4

Let $F(u) = f \circ u$, and let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Show that if $f \in C^{k+1}(\mathbb{R})$, then $F : C^k(\overline{\Omega}) \to C^k(\overline{\Omega})$, and F is C^1 with $dF(u)\xi = f'(u)\xi$.

Proof. □

5.5 Question 5

Theorem. 5.1: A Proposition

Let $\Omega \subset \mathbb{R}^n$. Let $F(u) = f \circ u$ with $f \in C^2(\Omega)$ and obeying the estimates:

$$|f^{(k)}(u)| \leq c|u|^{p-k} \quad \text{for } k = 0, 1, 2, \ldots$$

for some $p \geq 2$. Then $F : H^r(\Omega) \to L^2(\Omega)$, and is C^1, provided $r > n/2$.

Prove that, for $r > n/2$, under the conditions of the above Theorem:

1. *$F = f \circ u$ maps $H^r(\Omega)$ into $H^1(\Omega)$.*
2. *$F = f \circ u$ is Gâteaux differentiable, as a map from $H^r(\Omega)$ into $L^2(\Omega)$.*

Proof. □