APM346 HW 1

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1 Question 1

Given: $\rho_0(x, y) = e^{-x^2-y^2}$ which is the same as e^{-r^2} when doing a change of variables of $(r\cos(\theta), r\sin(\theta))$

1. Compute the flux over the boundary of the disc (noting that normal vector and radius are perpendicular):

$$\begin{aligned}\int_0^{2\pi} \left\langle \hat{J}(t, x, y), \hat{n} \right\rangle d\theta &= \int_0^{2\pi} \langle \alpha M_r(t)\hat{r}, \hat{n}\rangle \, d\theta \\ &= \alpha M_r(t) \int_0^{2\pi} \langle \hat{r}, \hat{n}\rangle \, d\theta \\ &= \alpha M_r(t) \int_0^{2\pi} d\theta \\ &= 2\alpha M_r(t)\pi\end{aligned}$$

2. Then we leverage this to solve for the change in the mass over time t which is equal to half of the initial mass of the disk (Recalling that $M_r(t) = \frac{M_r(0)}{2}$:

$$\begin{aligned}\frac{dM}{dt} &= 2\alpha M_r(t)\pi \\ \frac{dM}{M_r(t)} &= 2\alpha\pi dt \\ \int_0^t \frac{dM}{M_r(t)} &= \int_0^t 2\alpha\pi dt \\ ln(M_r(t)) - ln(M_r(0) &= 2\alpha\pi t \\ ln(2) &= 2\alpha\pi t \\ t &= \frac{ln(2)}{2\alpha\pi}\end{aligned}$$

Thus, the mass halves at time $t = \frac{ln(2)}{2\alpha\pi}$

2 Question 2

Setting up the equation now we have that

$$\begin{cases} \partial_t \rho + 10\partial_x \rho = 0 \\ \rho_0 = e^{-x} \end{cases}$$

We cam note that:
$\frac{dx}{dt} = 10m/s$, $I = [1, 2]$, and $t = 60$
From this we get:

$$x = 10t + x_0 \Rightarrow x_0 = x - 10t$$
$$\Rightarrow u = u_0(x - 10t) = u = e^{-x+10t}$$

And so by solving over I we get, and noting that $u(x, 60) = e^{-x+600}$:

$$\int_1^w e^{-x+600} dx = e^{600}[-e^{-x}]_1^2 = e^{600}(-e^{-2} + e^{-1}) = \frac{e^{601} - e^{600}}{e^2}$$

3 Question 3

$$\begin{cases} \partial_t u + c\partial_x u = 0 \\ u(0,x) = e^{-x^2} \end{cases}$$

$\frac{dx}{dt} = c \Rightarrow x = ct + x_0 \Rightarrow x - ct = x_0$

From the above equations we can tell that c is the rate of change of x in regards to t.

$u = u_0(x - ct) = e^{-(x-ct)^2}$

4 Question 4

$$\begin{cases} \partial_t u + x\partial_x u = 0 \\ u(0,x) = e^{-|x|} \end{cases}$$

$\frac{dx}{dt} = x \Rightarrow x = x_0 e^t \Rightarrow x_0 = xe^{-t}$ and so, since we have the solution such that $u_0(xe^{-t}) = u = e^{-1}$
We get that:

$$x_0 = \pm 1 \Rightarrow x = \pm e^t$$

5 Question 5

$$\begin{cases} \partial_t u + (1+x^2)\partial_x u = t \\ u(0,x) = x_0 \end{cases}$$

$\frac{dx}{dt} = 1 + x^2 \Rightarrow x = x_0 + \arctan(t) \Rightarrow x_0 = x - \arctan(t)$ we then use this to find that

$$\frac{dU}{dt} = t \Rightarrow u = \frac{t^2}{2} + u_0 = \frac{t^2}{2} + x_0 = \frac{t^2}{2} + x - \arctan(t)$$

And so:

$$u = \frac{t^2}{2} + x - \arctan(t)$$

6 Question 6

$$\begin{cases} \partial_t u + x\partial_x u = xe^{-t}u \\ u(0,x) = sin(x) \end{cases}$$

$\frac{dx}{dt} = x \Rightarrow x = x_0 e^t \Rightarrow x_0 = e^{-t}x$ we then can also solve $\frac{dU}{dt} = xe^{-t}u = x_0 u \Rightarrow u = u_0 e^{x_0 t}$
We can then use these two attributes to solve for u and get that $u = sin(xe^{-t})e^{xte^{-t}}$:

$$u = sin(x_0)e^{xt} = sin(xe^{-t})e^{xte^{-t}}$$

7 Question 7

$$\begin{cases} \partial_t u + \partial_x u = 2te^{-u} \\ u(0,x) = \ln(1+x^2) \end{cases}$$

$\frac{dx}{dt} = 1 \Rightarrow x_0 = x - t =$ we then use to find that

$$\frac{dU}{dt} = 2te^{-u} \Rightarrow e^u = t^2 + e^{u_0} \Rightarrow u = 2\ln(t) + u_0 = 2\ln(t) + \ln(1+x_0^2) = 2\ln(t) + \ln(1+(x-t)^2)$$

And so solution is the set M st.

$$n = \{(x,t) \in \mathbb{R}^2 | t \geq 0 \text{ and } 2\ln(t) + \ln(1+(x-t)^2) \leq e\}$$

8 Question 8

$$\begin{cases} \partial_t u + e^{u-x}\partial_x u = 0 \\ u(0,x) = x_0 \end{cases}$$

$\frac{dx}{dt} = e^{u-x} \Rightarrow e^x = e^u t + e^{x_0} \Rightarrow x - \ln(t)u = x_0$
And so solution is the set M st.

$$u = \frac{x}{1+\ln(t)}$$

9 Question 9

$$\begin{cases} \partial_t u + \sin(u)\partial_x u = 1 \\ u(0,x) = x \end{cases}$$

$\frac{dx}{dt} = \sin(u) \Rightarrow x = \sin(u)t + x_0 \Rightarrow x_0 = x - \sin(u)t$ we then use to find that

$$\frac{dU}{dt} = 1 \Rightarrow u = t + u_0 = t + x - \sin(u)t$$

And so:

$$u + t\sin(u) = t + x$$

Now solving for solution on interval $(0,t)$ we get:

$$F(x,t,u) = u + t\sin(u) - x - t$$

We can set $t = \frac{u}{1-\sin(u)}$ and get that

$$F(0, \frac{u}{1-\sin(u)}, u) = 0$$

Then we solve for the gradient of F and get:

$$\triangledown F(x,t,u) = (-1, \sin(u) - 1, 1 - t\cos(u))$$

And by solving for the gradient on the interval $(0,t)$ we get:

$$\triangledown F(0, \frac{u}{1-\sin(u)}, u) = (-1, \sin(u) - 1, 1 - t\cos(u))$$

And so, for a small enough t we can solve the above equation st. both $\triangledown F \neq 0$ and while $t = \frac{u}{1-\sin(u)}$ so that $F(x,t,u) = 0$

10 Question 10

A)

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ u(0,x) = & \begin{cases} 2 & x \leq 0 \\ 2-x & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases} \end{cases}$$

$\frac{dx}{dt} = u \Rightarrow x - ut = x_0 \Rightarrow u = u_0(x - ut)$

$$u = \begin{cases} 2 & x - ut \leq 0 \\ 2 - x + ut & 0 \leq x - ut \leq 1 \\ 1 & 1 \leq x - ut \end{cases}$$

$$\Rightarrow u = \begin{cases} 2 & x \leq 2t \\ \frac{2-x}{1-t} & 2t \leq x \leq 1+t \\ 1 & 1+t \leq x \end{cases}$$

Characteristic equations:

$$\begin{cases} 2t = x - x_0 & x \leq 2t \\ t = \frac{x-x_0}{2-x_0} & 2t \leq x \leq 1+t \\ t = x - x_0 & 1+t \leq x \end{cases}$$

We can also notice that $g(u) = \frac{1}{2}u^2$ and that $u_+ = 1, u_- = 2$ and so the shockwave spread is equal to 1.5 since:

$$v = \frac{g(1) - g(2)}{1-2} = \frac{0.5-2}{-1} = 1\frac{1}{2}$$

And the solution at $u(0.5,x), u(1,x), u(2,x)$ and $u(4,x)$ is:

11 Question 11

$$\begin{cases} \partial_t \rho + (1-2\rho)\partial_x \rho = 0 \\ \rho_0(x) & = \begin{cases} 0.25 & x < 0 \\ 0.25(x+1) & 0 < x < 1 \\ 0.5 & x > 1 \end{cases} \end{cases}$$

1. $\frac{dx}{dt} = 1 - 2\rho \Rightarrow x = t - 2t\rho + x_0 \Rightarrow x_0 = x - t + 2t\rho = x + t(2\rho - 1)$

2. Drawing out the solution of $\rho(0.5, x)$

$$\rho = \begin{cases} 0.25 & x < t(1-2\rho) \\ 0.25(x + t(2\rho - 1) + 1) & t(1-2\rho) < x < 1 + t(1-2\rho) \\ 0.5 & x > 1 + t(1-2\rho) \end{cases}$$

$$\rho = \begin{cases} 0.25 & x < \frac{1}{2}t \\ \frac{x-t+1}{4-2t} & \frac{1}{2}t < x < 1 + \frac{3}{2}t \\ 0.5 & x > 1 \end{cases}$$

$$\Rightarrow \rho(0.5, x) = \begin{cases} 0.25 & x < \frac{1}{4} \\ \frac{2x+1}{6} & \frac{1}{4} < x < 1 \\ 0.5 & x > 1 \end{cases}$$

3. let $g(\rho) = \rho - \rho^2$

$$\begin{aligned} v &= \frac{g(0.5) - g(0.25)}{0.5 - 0.25} \\ &= \frac{0.5 - 0.5^2 - 0.25 + 0.25^2}{0.25} \\ &= \frac{0.25^2}{0.25} \\ &= \frac{1}{4} \end{aligned}$$

And so the shockwave speed is $\frac{1}{4}$

12 Question 12

$$\frac{dx}{dt} = 1 \Rightarrow x = t + x_0 \Rightarrow x - t = x + 0$$

$$\frac{dy}{dt} = y \Rightarrow y = y_0 e^t \Rightarrow y_0 = ye^{-t}$$

$$\star\ u_0 = sin(ye^{-t}(x - t))$$

$$\star\star\ \frac{dU}{dt} = xy \Rightarrow u = xyt + u_0$$

$$\Rightarrow u = xyt + sin(ye^{-t}(x - t))$$

(By combining $\star$ and $\star\star$)